OPTIMAL CONSUMPTION AND PORTFOLIO CHOICE WITH NO-BORROWING CONSTRAINT IN THE KIM-OMBERG MODEL

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Abstract. We study an infinite-horizon optimal consumption and portfolio problem for an investor who receives a constant stream of labor income and faces a strict no-borrowing constraint. The financial market follows a Kim–Omberg type model in which the excess return of the risky asset is driven by a mean-reverting Ornstein–Uhlenbeck process, capturing the empirical evidence that expected excess returns are predictable and exhibit mean reversion. The investor allocates wealth between a risk-free asset and a risky asset while choosing a consumption plan so as to maximize expected discounted utility from consumption. The presence of labor income and the requirement that wealth remain nonnegative at all times prevent the use of standard homogeneity arguments typically employed in geometric-wealth settings and make the associated Hamilton–Jacobi–Bellman equation difficult to analyze directly.

To address these difficulties, we adopt a Lagrange duality approach that transforms the original constrained optimization problem into a dual singular stochastic control problem. The dynamic no-borrowing constraint is replaced by a static budget constraint involving a nonincreasing càdlàg process that acts as an endogenous Lagrange multiplier. In the dual formulation, the state variables consist of the dual process associated with marginal utility of wealth and the stochastic factor driving the excess return. Because the latter directly affects the diffusion coefficient of the dual state, the resulting control problem naturally features stochastic volatility.

A key step of our analysis is the probabilistic connection between the dual singular control problem and an auxiliary two-dimensional optimal stopping problem. The stopping problem involves a degenerate diffusion with stochastic volatility and yields a value function that determines the marginal value of wealth in the dual formulation. We show that the stopping region can be characterized by a free boundary depending on the stochastic excess-return factor. The value function of the optimal stopping problem is shown to be continuous and locally Lipschitz on the state space and infinitely differentiable inside the continuation region and the interior of the stopping region. Despite the degeneracy of the associated differential operator, we prove that Hörmander's condition holds, implying hypoellipticity and the existence of a smooth transition density for the state process. This regularity result allows us to establish continuous differentiability of the value function across the entire state space.

Using the link between optimal stopping and singular control, we fully characterize the optimal dual policy as the minimal reflection that keeps the dual state below the free boundary. Strong duality between the primal and dual problems is then established, allowing us to recover the investor's optimal consumption strategy, portfolio allocation, and wealth dynamics. Economically, the dual control admits a natural interpretation as a shadow price associated with the borrowing constraint: when wealth approaches zero, the marginal value of wealth increases and the dual state is reflected at the boundary, ensuring that the wealth process remains nonnegative. Finally, we provide numerical illustrations that highlight the impact of stochastic investment opportunities and the borrowing constraint on optimal consumption and portfolio policies.

Keywords: optimal consumption and portfolio choice, Kim-Omberg model, no-borrowing constraint, singular stochastic control, optimal stopping, stochastic volatility.

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