Matrix representations for prices of life-contingent derivatives

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Abstract

A life-contingent derivative is a derivative security that may be exercised at the time of a certain event. In the context of life-insurance, this event could be the death of a policyholder. At the time of death, the heirs will have the right to exercise the derivative security which will result in a certain (random) pay-out. Such arrangements are known as unit- or equity linked insurance contracts.

The papers [1] and [2] provide explicit formulas for the prices of a range of derivatives related to the life-insurance business, where the underlying models are Black-Scholes and (the exponential of) Lévy processes, respectively, and where the time until death is assumed to have a distribution whose density is a linear combination of exponentials.

We carry out an analysis similar to [1] and [2], assuming that the residual lifetime is phase-type distributed instead. We derive the joint distributions of a variety of objects of interest related to the sample path of a Brownian motion stopped at a phase-type time horizon, and we show how these expressions may be used for pricing a range of financial securities in a Black-Scholes market. The phase-type assumption offers several advantages, where the first is notational, since many formulas can be expressed explicitly in a more general context in terms of functions of matrices. The second advantage is generality, since we work with a much more flexible class of distributions than linear combinations of exponentials, such that the number of parameters needed will, in general, be lower for phase-type parameterization. Finally, phase-type distributions can easily be fitted to real data using a maximum likelihood approach readily available in R.

At the applied probabilistic level, there are several general novelties in the context of phase-type theory that may be of interest of their own. An example of this is based on the observation that if T is a sub-intensity matrix, then $-\sqrt{-T}$ is again a sub-intensity matrix. This leads to an explicit representation for a multivariate phase-type distribution describing the joint distribution of the running maximum and its distance from the current level of a Brownian motion stopped at a phase-type distributed time.

Keywords: Life-contingent derivatives, Stopped Brownian motion, Phase-type distributions.

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