Joint Implied Willow Tree: an approach for joint S&P 500/VIX Calibration

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Abstract

The Volatility Index, known as VIX, debuted in 1993 under the auspices of the Chicago Board Options Exchange (CBOE), initially formulated to gauge the market's anticipation of 30-day volatility inferred from at-the-money S&P 100 index option prices (VIX03). Since 2003, the VIX has undergone redefinition, now representing the square root sequence of a designated basket of options on the S&P 500 index (SPX) with a 30-day maturity. The coefficients of this basket are meticulously selected to ensure that at any given time t, the VIX reflects the annualized square root of the contract's price, where the contract payoff equals $\ln(S_{t+\tau}/S_t)$, with τ denoting 30 days and S_t representing the value of the SPX at time t. Therefore, it can be formally defined as

$$\text{VIX}_t^2(\tau) = -\frac{2}{\tau}\,\mathbb{E}_t^{\mathbb{Q}}\left[\ln\frac{S_{t+\tau}}{S_t e^{r\tau}}\right] \times 100^2, \tag{1}$$

where r is the risk free interest rate and $\mathbb{E}_t^{\mathbb{Q}}$ is the conditional expectation under the risk neutral measure $\mathbb{Q}$ given the market filtration $(\mathfrak{F}_t)_{t\geq 0}$. The introduction of the VIX option in 2006 added another dimension to volatility management tools for market participants. Since its inception, these products have swiftly emerged as some of the most liquid financial instruments globally.

Given that the VIX is derived from the SPX, any viable methodology must align with the pricing dynamics of SPX options. Consequently, numerous researchers and practitioners have endeavored to construct models that effectively calibrate to both SPX and VIX option prices. However, this task poses considerable challenges, particularly for short maturities. The pursuit of a solution to this joint calibration predicament has led to extensive examination within three distinct model categories. The first category involves employing parametric continuous diffusion models. Unfortunately, continuous diffusion models have not yet yielded complete success in this context. The second category of models incorporates jumps into the dynamics of the SPX.

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While jumps can reconcile the skewness of SPX options with VIX implied volatilities, these models still fall short of achieving satisfactory accuracy for joint calibration. Specifically, most of them struggle to replicate VIX smiles for maturities shorter than one month. It is evident that parametric models cannot perfectly encapsulate all SPX and VIX market dynamics. The final approach adopts a nonparametric discrete-time model, leveraging its complete flexibility for precise calibration. The pioneering work by [1] introduces the first nonparametric discrete-time model tailored for joint calibration, aiming for precise calibration of the VIX option smile at time T_1 and SPX option smiles at dates T_1 and $T_2 = T_1 + 30$ days. While theoretically extendable to any set of maturities, practical implementation poses significant challenges due to computational complexity.

In this study, we propose a novel nonparametric discrete time model called the Joint Implied Willow Tree (JIWT) to tackle the joint calibration challenge. Our method offers two primary contributions. Firstly, we delve into the conditional probability distributions between two maturities using joint SPX and VIX option prices. While single SPX option prices provide insight into SPX RNDs, they offer limited information on conditional densities. Leveraging the VIX definition (1) grounded in the SPX, we can identify conditional densities that align with VIX and its option prices. Secondly, our JIWT method is simpler, more straightforward to implement, and efficiently extends to multiple maturities of SPX and VIX options compared to the nonparametric discrete-time model proposed by [1]. This approach enables us to extract a more accurate Risk-Neutral Process (RNP) for jointly calibrating the volatility smile from SPX and VIX options. Numerical and empirical findings underscore the advantages of our novel RNP methodology in addressing the joint calibration problem.

Keywords: Joint implied willow tree, risk neutral process, S&P 500 index options, VIX options, Joint Calibration

References

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