Calibration of risk aversion to real pension asset allocation

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Abstract

We propose an approach to measure the risk aversion of an investor managing a diverse portfolio that includes pension savings, real estate, and free funds. Given a financial market, a wealth process, and a risk aversion parameter, Merton provides formulas for the optimal investment strategies [1, 2]. We utilise the investor's real asset allocation as the optimal strategy, and leveraging Merton's formulas, we infer the associated risk aversion parameter. To enhance the stability of this calibration, we construct a customised risky fund aligned with the investor's preferences. Finally, we study disparities of risk aversion across the financial categories of pension savings, real estate investment, and free funds for the investor by certainty equivalents. The results are reinforced by empirically calibrating the risk aversions of a real Danish pension portfolio in a numerical study.

Individual preferences are often articulated through a utility function, representing the investor's risk aversion and attitude towards risk. The investor's risk aversion is a fundamental element in financial decision-making but lacks a standardised calibration method. We assume that the investor's preferences adhere to a power utility function. This is also known as the Constant Relative Risk Aversion (CRRA) utility function given by

$$F(t,x) = \frac{1}{1-\gamma} e^{-\rho t} x^{1-\gamma},$$

where $\gamma \in (0,\infty)\backslash\{1\}$ is a coefficient of relative risk aversion, and $\rho > 0$ is an impatience factor.

We use stochastic control theory and the Bellman principle of optimality to obtain an optimal investment portfolio based on the CRRA utility function. The optimal investment problem in a continuous-time model is first presented in [1, 2]. For the pension savings, we account for the present value of future premiums, which results in an optimal investment strategy consistent

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with real life-cycle pension products as described in [3], [4] and [5]. The optimal investment in n risky asset is

$$\mathbf{w}^*(t,x) = \mathbf{\Sigma}^{-1}(\alpha - r\mathbf{e})\frac{1}{\gamma}\frac{x + h(t)}{x}, \quad (1)$$

where $\mathbf{\Sigma}$ is the variance matrix, $\boldsymbol{\alpha}$ is the expected returns, r is the risk-free short rate and $h(x)$ is the present value of future premiums. This gives us a relation between risk aversion and an optimal investment. Our approach reverses the perspective to calibrate the risk aversion from Equation (1). We assume the investor reveals their attitude towards risk in their real choice of financial allocation. We thereby use the investor's real asset allocation to measure their implicit risk aversion, assuming that the allocation is optimal for the investor.

Each position in risky assets constitutes an equation for risk aversion, making it an overdetermined system. A linear solution to the overdetermined system is unstable and yields solutions different from what has previously been estimated. To enhance the stability of the calibration, we construct a risky fund aligned with the investor's preferences. Typically, the investor chooses an investment strategy predefined by a pension provider for pension savings, which amounts to one risky fund. Therefore, we consider a setup where the investor allocates the wealth between the risk-free money market account and a mutual risky fund the insurance company has constructed in such a way that it is optimal for the investor to invest their wealth in the fund. This results in a more reliable measure of risk aversion given by

$$\hat{\gamma} = \frac{1}{\hat{\pi}(t, X(t))}\frac{\alpha_\theta(t, X(t)) - r}{\sigma_\theta(t, X(t))^2}\frac{X(t) + h(t)}{X(t)}. \quad (2)$$

where $\alpha_\theta(t, X(t))$ is the expected return and $\sigma_\theta(t, X(t))$ is the volatility of the fund.

The three categories of pension savings, real estate investment, and investment of free funds cover typical exposures to financial risk. Across the financial categories, we obtain risk aversion estimates, and potential disparities in the estimates may express inconsistency in the investor's behaviour. Specifying the difference in terms of the value of the risk aversion parameters is not explanatory to the investor. Therefore, we express the difference in terms of certainty equivalence, translating utility to a monetary scale. Investors can then compare themselves to peers and gain insight from certainty equivalents.

The numerical study considers 5236 policies of the product *P+ Livscyklus* from the Danish pension provider *P+* in 2022, where the deferred life annuity is distributed over the low, moderate, and high-risk investment profiles. The calibrated risk aversion parameters are in the range of 2 to 10.

Keywords: Risk aversion; Risk aversion preference; Stochastic control; Personal finance; Certainty equivalents.

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