On geometrically convex risk measures

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Abstract

Geometrically convex functions, also known in the literature as log-log convex or GG-convex functions, are positive functions of a positive real variable satisfying $f(x^\lambda \cdot y^{1-\lambda}) \leq f^\lambda(x) \cdot f^{1-\lambda}(y)$, for each $x, y > 0$ and $\lambda \in (0,1)$. Our first contribution is the introduction of a notion of GG-convex conjugate defined as follows:

$$f^\diamond(y) := \sup_{x>0} \left\{ \frac{\exp(\log x \log y)}{f(x)} \right\}.$$

GG-convex conjugation is a multiplicative version of the Fenchel transform and has similar properties, such as being order-reversing in the sense that $f \leq g \Rightarrow f^\diamond \geq g^\diamond$, involutive in the sense that under suitable assumptions $f^{\diamond\diamond} = f$, and multiplicative with respect to multiplicative inf-convolution in the sense that $(f \otimes g)^\diamond = f^\diamond \cdot g^\diamond$, where $(f \otimes g)(z) := \inf_{x,y>0,\, xy=z} \{f(x) \cdot g(y)\}$. Remarkably, we show that these properties form the basis of of two possible axiomatizations of GG-convex transform, following the spirit of [1, 2], and indeed we show that GG-convex conjugation is a “general duality transform" in the sense of [2].

We then move to the study of GG-convex risk measures, similarly defined as GG-convex functionals on general model spaces as in [6]. The relevance of the notion of GG-convexity for the study of return risk measures has been noted in several recent papers (see e.g., [3, 5]). We show how the notion of GG-convex conjugate provided for real functions of real variable leads to the following general dual representation of GG-convex risk measures satisfying a version of the Fatou property

$$\rho(X) = \sup_{Y \in \mathcal{X}^*_{\log}} \left\{ \frac{\exp(\mathbb{E}[\mathrm{logYlogX}])}{\rho^\diamond(Y)} \right\},$$

where $\rho^\diamond$ is the GG-convex conjugate defined as in the scalar case.

We show how related dual representations given in [3] under the additional assumptions of positive homogeneity and monotonicity can be derived as special cases.

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Finally, we focus on the law-invariant case and study consistency properties of law-invariant GG-convex risk measures with respect to stochastic orders, paralleling the classical results of [4]. To this aim, we introduce the multiplicative version of the convex order and of the increasing convex order as follows:

$$X \leq_{\mathrm{GA-cx}} Y \text{ if } \log X \leq_{\mathrm{cx}} \log Y$$
$$X \leq_{\mathrm{GA-icx}} Y \text{ if } \log X \leq_{\mathrm{icx}} \log Y$$

and we show that if ρ is law-invariant and GG-convex then

$$X \leq_{\mathrm{GA-cx}} Y \Rightarrow \rho(X) \leq \rho(Y)$$

while if ρ is also monotone then

$$X \leq_{\mathrm{GA-icx}} Y \Rightarrow \rho(X) \leq \rho(Y).$$

Keywords: Geometric convexity, duality transforms, risk measures, dual representations, stochastic orders.

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