On the relation between discrete and continuous-time affine option pricing models

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Extended Abstract

This paper derives the weak limits of a general class of affine GARCH pricing models which feature stochastic volatility, and examines the impact of using different convergence rates for the underlying model parameters on option prices.

Our paper makes three important contributions to the literature. *First*, we derive novel diffusion limits for affine GARCH component pricing models based on both Gaussian and inverse Gaussian innovations, and risk-neutralized with a variance dependent pricing kernel. To achieve this, we introduce a general discrete-time framework which nests both model specifications, and which allows us to use a unique set of convergence rate assumptions on the model parameters which do not depend on the choice of the underlying distribution. These rates align with those proposed by Nelson (1990) and Duan (1997) for establishing weak limits across a large plethora of non-affine GARCH models. The first convergence result show that the weak limits of Gaussian and inverse Gaussian GARCH component models under the physical measure correspond to three-dimensional affine degenerate stochastic volatility models. In particular, the short and long-run variance components are driven by the same two independent Brownian motions, one of which also drives the return process. Another important finding is that the diffusion limits of the Gaussian and inverse Gaussian coincide if and only if specific relationship between the two model parameters hold. Interestingly, these relationships are consistent with those found in Babaoğlu et al. (2018) to ensure an entirely different type of convergence of inverse Gaussian GARCH models to their Gaussian counterparts as the skewness parameter of the former distribution approaches zero. The second main convergence result identify the weak limits of these models under the variance dependent pricing kernel. This is carried through an additional assumption on the convergence rate of the variance risk-preference parameter from the pricing kernel, which ensures the convergence of the Radon-Nycodim derivatives. The resulting limit is also a three-dimensional degenerate diffusion which can be recovered directly by applying Girsanov's theorem to the physical limit for particular choices of the market prices of risk associated to the Brownian motions governing the variance component dynamics.

Our *second* theoretical contribution relates to the investigation of the use of different convergence rates for the underlying model parameters on the diffusion limit. Our first result shows that when restricted to

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a one-component, the limit of the Gaussian GARCH model obtained above generalizes the Heston and Nandi (2000) result. Specifically, our limit is a two-dimensional non-degenerate affine diffusion driven by two stochastically correlated Brownian motions, instead of a degenerate diffusion driven by perfectly (negatively) correlated Brownian motions. This is a direct consequence of using different convergence rates and, to our knowledge, our paper is the first that brings this disparity to light for affine models. We contend that the rates that were previously considered do not lead to the most general weak diffusion limit dynamics and, in fact, lead to diffusive dynamics that exhibit some inconsistency with the statistical properties of the underlying discrete-time process. Motivated by this finding, and aiming to better understand the link between discrete and continuous-time pricing models, we provide a full classification of all possible limits for two categories of GARCH models which exhibit stochastic volatility: the first class of models has been recently proposed by Heston et al. (2023), to relax the GARCH assumption of the Heston and Nandi (2000) model by incorporating a different noise process into the variance dynamics, which is correlated to the returns innovation, while maintaining the affine property; the second class consists of a generalization of the non-affine square-root GARCH models proposed by Ishida and Engle (2002) to allow for stochastic volatility. Our results suggest that for both models, and regardless of the convergence rate used, we obtain affine weak diffusion limits. Moreover, our convergence rates provide the most complex dynamics which match the moment structure of their discrete-time counterparts.

Our *third* contribution relates to numerically examining the effect of using alternative convergence rates on European call option prices computed based on the model of Heston et al. (2023). We find that our convergence rates provide the smallest difference between option prices computed based on discrete-time models and on their diffusion limits. The sensitivity of these prices relative to the shape of the pricing kernel (monotonic vs non-monotonic), the correlation between the innovations driving the returns and conditional variance processes, and the magnitude of the leverage parameter, are further examined. Our results suggest that both the non-monotonic pricing kernel and the correlation coefficient affect the prices significantly, and our rates provide the "best" limiting prices in the sense they are the closest to their discrete-time counterparts.

Keywords: Affine models; multi-component volatility; convergence rates; weak diffusion limits; exponential affine pricing kernel

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