Perpetual American Options in Two-Dimensional Diffusion Models

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Abstract

We study the perpetual American options optimal stopping problems

$$V(x,y) = \sup_{\tau} \mathbb{E}_{x,y}\big[e^{-r\tau}\, G(X_\tau, Y_\tau)\big] \tag{1}$$

associated with the gain (or payoff) functions

$$G(x,y) = K - L\,y - x \quad \text{and} \quad G(x,y) = (K-x)^+\,(L-y)^+ \tag{2}$$

where $r>0$ and $K, L > 0$ are some given constants. We assume that the components of the two-dimensional process $(X,Y) = (X_t, Y_t)_{t\geq 0}$ solve the stochastic differential equations

$$dX_t = (r-\delta_1)\,X_t\,dt + \sigma_1\,X_t\,dB^1_t \quad (X_0 = x) \tag{3}$$

$$dY_t = (r-\delta_2)\,Y_t\,dt + \sigma_2\,Y_t\,dB^2_t \quad (Y_0 = y) \tag{4}$$

for some given constants $r>0$, $\delta_j \geq 0$ and $\sigma_j > 0$, for $j=1,2$. Here $B^j = (B^j_t)_{t\geq 0}$, for $j=1,2$, are standard Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ started at 0 and constantly correlated with $\mathbb{E}[B^1_t B^2_t] = \rho t$ for all $t \geq 0$ and some $\rho \in (-1,1)$ given fixed. It is assumed that the expectation in (1) is taken with respect to the probability measure $\mathbb{P}_{x,y}$ under which the process (X,Y) starts at some point $(x,y) \in (0,\infty)^2$, and the supremum in (1) is taken over all stopping times τ with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of (X,Y).

McDonald and Siegel [5] solved the problem of pricing perpetual American exchange option with the payoff function $G(x,y) = x - Ky$ in the framework of the two-dimensional model defined in (3)-(4). Olsen and Stensland [7] and Hu and Øksendal [3] provided sufficient and necessary conditions for the optimal exercise of the exchange options in the multi-dimensional version of the model presented in (3)-(4). Nishide and Rogers [6] extended the results of the previous papers to the case of more complicated exchange options on several underlying assets. The problems of pricing of American-type options on several assets and the related multi-dimensional optimal stopping problems were also studied in Jaillet, Lamberton, and Lapeyre [4], Broadie and Detemple [1], and Villeneuve [8] among others, for continuous diffusion models.

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In the present paper, we consider perpetual American options with payoffs defined in (2). The resulting optimal stopping problems in (1) are necessarily two-dimensional in the sense that they cannot be reduced to optimal stopping problems for one-dimensional continuous Markov processes. In these cases, the optimal exercise boundaries for one risky asset price can be expressed as functions of the current state of the other. We find closed formulas for the value function expressed in terms of the optimal stopping boundaries which in turn are shown to be unique solutions to nonlinear Fredholm integral equations. A key argument in the existence proof is played by a pointwise maximisation of the expression obtained by the change-of-measure arguments. This provides tight bounds on the optimal stopping boundaries as well as describes their shape and asymptotic behaviour for small coordinate values of X and Y. The corresponding tight bounds for some non-discounted essentially two-dimensional optimal stopping problems have been recently derived in [2].

Keywords: Two-dimensional diffusion process, perpetual American options with additive and multiplicative payoffs, elliptic partial differential free-boundary problem, nonlinear Fredholm integral equation for the exercise boundary, the change-of-variable formula with local time on curves/surfaces.

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