Deep Learning the Parameters of Fractional Processes and Assessing Their Accuracy

László Márkus[1], **Dániel Boros**[1]

[1]Department of Probability Theory and Statistics, Institute of Mathematics Eötvös Loránd University, Budapest, Hungary
e-mail: laszlo.markus@ttk.elte.hu

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This research explores the reliability of deep learning, specifically Long Short-Term Memory (LSTM) networks, for estimating the Hurst parameter in fractional stochastic processes. The study focuses on two types of processes: the fractional Brownian motion (fBm) and the fractional Ornstein-Uhlenbeck (fOU) process. The work involves a fast generation of extensive datasets for fBm and fOU to train the LSTM network on a large volume of data in a feasible time. The study analyses the accuracy of the LSTM network's Hurst parameter estimation regarding various performance measures like RMSE, MAE, MRE, and quantiles of the absolute and relative errors. It finds that LSTM outperforms the traditional statistical methods, the relative error, however, is still significant. The research also delves into the implications of training length and valuation sequence length on the LSTM's performance.

A **(scaled) fractional Brownian motion (fBm)**, denoted as $B_H(t)$, is a Gaussian process initiating from zero and characterized by continuous time, zero mean, and the autocovariance function $E\left[B_H(t) \cdot B_H(s)\right] = \frac{\sigma^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$. The fBm is notable for having stationary and **dependent** increments and is recognized for being a **self-similar** process. The Hurst exponent H and the scale σ are its parameters.
A mean-reverting **fractional Ornstein-Uhlenbeck (fOU)** process, also known in the finance literature as the fractional Vasicek model, is defined by a mean reverting fractional stochastic Langevin differential equation $dX(t) = \kappa(\theta - X(t))dt + \sigma dB_H(t)$, where the process is driven by a standard fBm $B_H(t)$ with the **Hurst** exponent $H \in (0,1)$, and $\kappa, \sigma > 0,\ \theta \in \mathbb{R}$.

The network architecture used in our experiments is composed of a two-layer unidirectional LSTM with an input dimension of 1 and an inner representation dimension of 128. The MLP, utilized in both models, has three layers with output dimensions of 128, 64, and 1, respectively, and incorporates a PReLU activation function between its first two layers. For model training, we employed AdamW optimization targeting the MSE loss function. The learning rate was established at 10^{-4}, with the batch size for both training and validation set at 32. The training spans 100 epochs with each epoch generating 100,000 sequences/trajectories. The trained network is then used to estimate the Hurst parameter of 10000 fBm and fOU trajectories of length 400, 1600, and 6400, respectively. We evaluate the estimation in terms of the RMSE, the MAE, and the mean relative errors (MRE). Rarely occurring severe errors can be very much intolerable in some applications. Therefore, beyond the mean error values, we also calculate the 95% quantiles and the medians of the absolute and relative errors.

Type	Evaluation length	RMSE	MAE	MRE%	Absolute Error		Relative Error %	
					q95%	q50%	q95%	q50%
LSTM on fBm	400	0.0311	0.0241	7.32	0.0630	0.0194	24.45	4.66
	1600	0.0149	0.0117	3.63	0.0300	0.0094	11.73	2.21
	6400	0.0079	0.0066	2.43	0.0165	0.0056	9.80	1.24
LSTM on fOU	400	0.0295	0.0229	7.02	0.0599	0.0184	22.53	4.33
	1600	0.0157	0.0122	3.62	0.0321	0.0098	11.51	2.31
	6400	0.0107	0.0085	2.28	0.0233	0.0063	6.65	1.60

Table 1: Performance metrics for LSTM evaluated on fBm and fOU. The LSTM network was trained on 1600-long samples.

It is more difficult to estimate the drift parameter κ of the fOU process. The full evaluation of the LSTM estimate is still ongoing work, but in Figure 1. the superior performance of LSTM (2nd and 3rd rows) to the state-of-the-art statistical estimator (first row) is apparent.

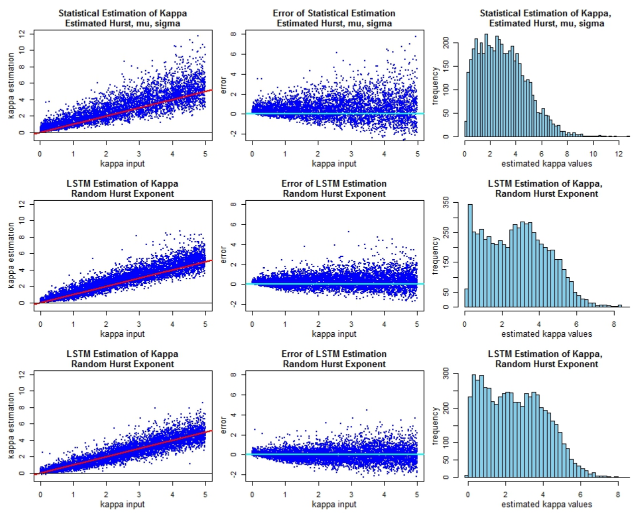

Figure 1: The Hurst parameter's learning quality evaluation of a fOU-1600 trained LSTM estimator when it is applied to 10000 unit drift fOU processes, observed in variable resolution with mesh sizes 400, 1600, 6400, and 25600.

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