Title: Implied Volatility, Options models and mimicking.

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Abstract

1. Implied volatility

Implied volatility is the volatility of the asset implied by option prices. Dupire's formula is a popular and powerful tool used in mathematical finance to reconstruct the volatility function from market prices of European call options. Dupire [2], [3] assumed that the stock price X_t is a diffusion $dX_t = \sigma(X_t, t)dB_t$, for some deterministic volatility function $\sigma(x, t)$, and a Brownian motion B_t. He gave the following formula (without proof). If $C(x, t)$ denotes the arbitrage-free price of a call option with strike x and expiration t, and assuming a zero riskless interest rate, $\sigma^2(x, t) = 2\frac{\frac{\partial}{\partial t}C(x,t)}{\frac{\partial^2}{\partial x^2}C(x,t)}$. [10] derived a PDE for $C(x, t)$ when X_t is a continuous martingale $\frac{\partial}{\partial t}C(x, t) = \frac{1}{2}h(x, t)^2 \frac{\partial^2}{\partial x^2}C(x, t)$, where $h(x, t) = \sqrt{\mathbb{E}\left(\frac{d\langle X,X\rangle_t}{dt}\middle| X_t = x\right)}$ is known as the local volatility. In the special case where X_t is a diffusion Dupire's formula immediately follows. [5] give Dupire's formula under the weakest (to date) assumptions.

2. Implied volatility in options models.

Can you have a model in which options prices are given, say by Black-Scholes formula, but the volatility is not constant, some function $\sigma(x, t)$? The answer, essentially, is no, [6], [8]. Let Z_t be a positive process and $dZ_t = \sigma h(t)\beta(Z_t)dB_t$, where h and β are deterministic functions, and $h \neq 0$. Denote $C(T, t, K, \sigma, z) = \mathbb{E}[(Z_T - K)^+|Z_t = z]$ prices of call options. Let S_t be another process with option prices that coincide with these on Z, for all strikes K and 3 maturities $T_1 < T_2 < T_3$ and all t, then processes S and Z are same.

Theorem 1 *Let S_t and θ_t be adapted processes such that $\theta_0 = \sigma$ and $S_0 = z_0$. Assume that S_t is non-negative and that there exist three terminal times, $T_1 < T_2 < T_3$ such that, for all $K \geq 0$ and all $t \leq T_i$, $i = 1, 2, 3$*

$$\mathbb{E}[(S_{T_i} - K)^+|\mathcal{F}_t] = C(T_i, t, K, \theta_t, S_t). \quad (1)$$

Then $\theta_t^2 = \sigma^2$ for all $t \leq T_1$, and $(S_t)_{t\leq T_1} =^d (Z_t)_{t\leq T_1}$. In other words Z is the only model on a Brownian filtration that is compatible with (1).

3. Mimicking

If we drop the requirement that options prices formulae hold for all t, but (1) only holds true for $t = 0$, for all strikes and all maturities, then all marginal distributions of S are fixed. Then there are many processes with same marginal

distributions (mimicking or faking). Martingales with given marginal. Classical results: Strassen 1965, Kellerer 1972, Gyongy (diffusions), Hirsch, Profeta, Roynette, Yor, 2011: Peacocks and Associated martingales, 2011.

Mimicking Brownian motion. Randomizing correlation. [7] A Family of Non-Gaussian Martingales with Gaussian Marginals. These are not continuous processes. [1] constructed continuous martingale, and [11] gave a simple construction and called it fake BM. [4] mimicked self-similar processes by randomizing the transition function. [9] mimic martingale diffusion by distribution mixing thus achieving a fake exponential BM.

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