Self-similar Markov processes
Part II: higher dimensions

Andreas Kyprianou
University of Bath

A more thorough set of lecture notes can be found here:
https://arxiv.org/abs/1707.04343
Other related material found here
https://arxiv.org/abs/1511.06356
https://arxiv.org/abs/1706.09924
PART I: ONE DIMENSION

▶ §1. Quick review of Lévy processes
▶ §2. Self-similar Markov processes
▶ §3. Lamperti Transform
▶ §4. Positive self-similar Markov processes
▶ §5. Entrance Laws
▶ §6. Real valued self-similar Markov processes

PART II: HIGHER DIMENSIONS

▶ §7. Isotropic stable processes in dimension $d \geq 2$ seen as Lévy processes
▶ §8. Isotropic stable processes in dimension $d \geq 2$ seen as a self-similar Markov process
▶ §9. Riesz–Bogdan–Żak transform
▶ §10. Hitting spheres
▶ §11. Spherical hitting distribution
▶ §12. Spherical entrance/exit distribution
§7. Isotropic stable processes in dimension $d \geq 2$ seen as Lévy processes
For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a $d$-dimensional isotropic stable process.

- $X$ has stationary and independent increments (it is a Lévy process)
ISOTROPIC $\alpha$-STABLE PROCESS IN DIMENSION $d \geq 2$

For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a $d$-dimensional isotropic stable process.

- $X$ has stationary and independent increments (it is a Lévy process)
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(e^{i\theta \cdot X_1})$ satisfies

\[ \Psi(\theta) = |\theta|^\alpha, \quad \theta \in \mathbb{R}. \]
Isotropic $\alpha$-stable process in dimension $d \geq 2$

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- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
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- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,
  \[ \Pi(B) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(-\alpha/2)} \int_B \frac{1}{|y|^{\alpha+d}} \, dy \]
  \[ = \frac{2^{\alpha-1} \Gamma((d + \alpha)/2) \Gamma(d/2)}{\pi^d \Gamma(-\alpha/2)} \int_{S_{d-1}} r^{d-1} \sigma_1(d\theta) \int_0^\infty 1_B(r\theta) \frac{1}{r^{\alpha+d}} \, dr, \]
  where $\sigma_1(d\theta)$ is the surface measure on $S_{d-1}$ normalised to have unit mass.
- $X$ is Markovian with probabilities denoted by $\mathbb{P}_x, x \in \mathbb{R}^d$.
ISOTROPIC $\alpha$-STABLE PROCESS IN DIMENSION $d \geq 2$

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- $X$ is Markovian with probabilities denoted by $\mathbb{P}_x, x \in \mathbb{R}^d$
Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$, under $\mathbb{P}_x$, the law of $(cX_{\sqrt{t}} - \alpha \alpha t, t \geq 0)$ is equal to $\mathbb{P}_{cx}$. 

ISOTROPIC $\alpha$-STABLE PROCESS IN DIMENSION $d \geq 2$
Isotropic $\alpha$-stable process in dimension $d \geq 2$

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▶ Isotropy means, for all orthogonal transformations (e.g. rotations) $U : \mathbb{R}^d \to \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

under $\mathbb{P}_x$, the law of $(UX_t, t \geq 0)$ is equal to $\mathbb{P}_{UX}$. 
ISOTROPIC $\alpha$-STABLE PROCESS IN DIMENSION $d \geq 2$

- Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,
  
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- If $(S_t, t \geq 0)$ is a stable subordinator with index $\alpha/2$ (a Lévy process with Laplace
  exponent $-t^{-1} \log \mathbb{E}[e^{-\lambda S_t}] = \lambda^\alpha$) and $(B_t, t \geq 0)$ for a standard (isotropic)
  $d$-dimensional Brownian motion, then it is known that $X_t := \sqrt{2}B_{S_t}$, $t \geq 0$, is a
  stable process with index $\alpha$.

  $$\mathbb{E}[e^{i\theta X_t}] = \mathbb{E} \left[ e^{-\theta^2 S_t} \right] = e^{-|\theta|^\alpha t}, \quad \theta \in \mathbb{R}. $$
SAMPLE PATH, $\alpha = 1.9$
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Some classical properties: Transience

We are interested in the potential measure

\[ U(x, dy) = \int_0^\infty \mathbb{P}_x(X_t \in dy) dt = \left( \int_0^\infty p_t(y-x) dt \right) dy, \quad x, y \in \mathbb{R}. \]

Note: stationary and independent increments means that it suffices to consider
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**Theorem**
The potential of \( X \) is absolutely continuous with respect to Lebesgue measure, in which case, its density in collaboration with spatial homogeneity satisfies \( U(x, dy) = u(y - x)dy, x, y \in \mathbb{R}^d, \) where

\[ u(z) = 2^{-\alpha} \pi^{-d/2} \frac{\Gamma((d - \alpha)/2)}{\Gamma(\alpha/2)} |z|^{\alpha - d}, \quad z \in \mathbb{R}^d. \]
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\]

In this respect \(X\) is transient. It can be shown moreover that

\[
\lim_{t \to \infty} |X_t| = \infty
\]

almost surely.
Proof of Theorem

Now note that, for bounded and measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$
\mathbb{E} \left[ \int_0^\infty f(X_t)dt \right] = \mathbb{E} \left[ \int_0^\infty f(\sqrt{2}B_S)dt \right] \\
= \int_0^\infty ds \int_0^\infty dt \mathbb{P}(S_t \in ds) \int_\mathbb{R} \mathbb{P}(B_s \in dx)f(\sqrt{2}x) \\
= \frac{1}{\Gamma(\alpha/2)\pi^{d/2}2^d} \int_\mathbb{R} dy \int_0^\infty ds e^{-|y|^2/4s} s^{-(\alpha-d)/2}f(y) \\
= \frac{1}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_\mathbb{R} dy |y|^{(\alpha-d)} \int_0^\infty du e^{-u} u^{-(d-\alpha/2)}f(y) \\
= \frac{\Gamma((d-\alpha)/2)}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_\mathbb{R} dy |y|^{(\alpha-d)}f(y).
$$
Some classical properties: Polarity

- Kesten-Bretagnolle integral test, in dimension $d \geq 2$,

$$\int_{\mathbb{R}} \text{Re} \left( \frac{1}{1 + \Psi(z)} \right) dz = \int_{\mathbb{R}} \frac{1}{1 + |z|^\alpha} dz \propto \int_{\mathbb{R}} \frac{1}{1 + r^\alpha} r^{d-1} dr \, \sigma_1(d\theta) = \infty.$$

- $\mathbb{P}_x(\tau\{y\} < \infty) = 0$, for $x, y \in \mathbb{R}^d$.

- i.e. the stable process cannot hit individual points almost surely.
§8. Isotropic stable processes in dimension $d \geq 2$ seen as a self-similar Markov process
Lamperti-transform of $|X|$  

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic $d$-dimensional stable process, the underlying Lévy process, $\xi$ that appears through the Lamperti has characteristic exponent given by

$$
\Psi(z) = 2^\alpha \frac{\Gamma\left(\frac{1}{2}(-iz + \alpha)\right)}{\Gamma\left(-\frac{1}{2}iz\right)} \frac{\Gamma\left(\frac{1}{2}(iz + d)\right)}{\Gamma\left(\frac{1}{2}(iz + d - \alpha)\right)}, \quad z \in \mathbb{R}.
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Here are some facts that can be deduced from the above Theorem that are exercises in the tutorial:
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Here are some facts that can be deduced from the above Theorem that are exercises in the tutorial:

- The fact that $\lim_{t \to \infty} |X_t| = \infty$
- The fact that $|X_t|^\alpha - d$, $t \geq 0$, is a martingale.
We can define the change of measure

\[
\frac{dP^0_x}{dP_x} \bigg|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0, x \neq 0
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We can define the change of measure
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\frac{d\mathbb{P}^0_x}{d\mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0, x \neq 0
\]

Suppose that \( f \) is a bounded measurable function then, for all \( c > 0 \),
\[
\mathbb{E}^0_x[f(cX_{c^{-\alpha}s}, s \leq t)] = \mathbb{E}_x \left[ \frac{|cX_{c^{-\alpha}t}|^{\alpha-d}}{|cx|^{d-\alpha}} f(cX_{c^{-\alpha}s}, s \leq t) \right]
= \mathbb{E}_{cx} \left[ \frac{|X_t|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t) \right] = \mathbb{E}^0_{cx}[f(X_s, s \leq t)]
We can define the change of measure

\[
\frac{d\mathbb{P}_x^\circ}{d\mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0, x \neq 0
\]

Suppose that \(f\) is a bounded measurable function then, for all \(c > 0,\)

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\]

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= \mathbb{E}_{cx} \left[ \frac{|X_s|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t) \right] = \mathbb{E}_{cx}^\circ[f(X_s, s \leq t)]
\]

Markovian, isotropy and self-similarity properties pass through to \((X, \mathbb{P}_x^\circ), x \neq 0.\)
CONDITIONED STABLE PROCESS

- We can define the change of measure

\[
\frac{d\mathbb{P}_{X}^{\circ}}{d\mathbb{P}_{X}}\bigg|_{\mathcal{F}_{t}} = \frac{|X_{t}|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0, x \neq 0
\]

- Suppose that \( f \) is a bounded measurable function then, for all \( c > 0 \),

\[
\mathbb{E}_{X}^{\circ}[f(cX_{c-\alpha s}, s \leq t)] = \mathbb{E}_{X}\left[\frac{|cX_{c-\alpha t}|^{\alpha-d}}{|cX|^{d-\alpha}}f(cX_{c-\alpha s}, s \leq t)\right] = \mathbb{E}_{cx}\left[\frac{|X_{t}|^{\alpha-d}}{|X|^{d-\alpha}}f(X_{s}, s \leq t)\right] = \mathbb{E}_{cx}^{\circ}\left[f(X_{s}, s \leq t)\right]
\]

- Markovian, isotropy and self-similarity properties pass through to \((X, \mathbb{P}_{X}^{\circ}), x \neq 0\).

- Similarly \((|X|, \mathbb{P}_{X}^{\circ}), x \neq 0\) is a positive self-similar Markov process.
It turns out that $(X, \mathbb{P}_x), x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.
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More precisely, for \(A \in \sigma(X_s, s \leq t)\), if we set \(\{0\}\) to be ‘cemetery’ state and \(k = \inf\{t > 0 : X_t = 0\}\), then

\[
\mathbb{P}^\circ_x (A, t < k) = \lim_{a \downarrow 0} \mathbb{P}_x (A, t < k|\tau_a^\oplus < \infty),
\]

where \(\tau_a^\oplus = \inf\{t > 0 : |X_t| < a\}\).
It turns out that $(X, \mathbb{P}_X^x), x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.

More precisely, for $A \in \sigma(X_s, s \leq t)$, if we set $\{0\}$ to be 'cemetery' state and $k = \inf \{t > 0 : X_t = 0\}$, then

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where $\tau_a^+ = \inf \{t > 0 : |X_t| < a\}$.

In light of the associated Esscher transform on $\xi$, we note that the Lamperti transform of $(|X|, \mathbb{P}_X^x), x \neq 0$, corresponds to the Lévy process with characteristic exponent

$$\Psi^o(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-iz + d))}{\Gamma(-\frac{1}{2}(iz + \alpha - d))} \frac{\Gamma(\frac{1}{2}(iz + \alpha))}{\Gamma(\frac{1}{2}iz)}, \quad z \in \mathbb{R}.$$
It turns out that \((X, \mathbb{P}_X^0), x \neq 0\), corresponds to the stable process conditioned to be continuously absorbed at the origin.

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\[
\mathbb{P}_x^0(A, t < k) = \lim_{a \downarrow 0} \mathbb{P}_x(A, t < k | \tau_{a^\oplus} < \infty),
\]

where \(\tau_{a^\oplus} = \inf\{t > 0 : |X_t| < a\}\).

In light of the associated Esscher transform on \(\xi\), we note that the Lamperti transform of \((|X|, \mathbb{P}_X^0), x \neq 0\), corresponds to the Lévy process with characteristic exponent

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\]

Given the pathwise interpretation of \((X, \mathbb{P}_X^0), x \neq 0\), it follows immediately that \(\lim_{t \to \infty} \xi_t = -\infty, \mathbb{P}_x^0\) almost surely, for any \(x \neq 0\).
**Definition**

A $\mathbb{R}^d$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a $\mathbb{R}^d$-valued self-similar Markov process if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$
\text{the law of } (cZ_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx},
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where $P_x$ is the law of $Z$ when issued from $x$. 

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**$\mathbb{R}^d$-self-similar Markov processes**
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▶ Same definition as before except process now lives on \(\mathbb{R}^d\).
**R^d**-self-similar Markov processes

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where \( P_x \) is the law of \( Z \) when issued from \( x \).

- Same definition as before except process now lives on **R^d**.
- Is there an analogue of the Lamperti representation?
In order to introduce the analogue of the Lamperti transform in $d$-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

**Definition**

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $P_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a Markov additive process (MAP) if $\Theta$ is a regular Feller process on $E$ with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

$$E_{x,\theta}[f(\xi_{t+s} - \xi_t, \Theta_{t+s})|\sigma((\xi_u, \Theta_u), u \leq t)] = E_{0,\Theta}[f(\xi_s, \Theta_s)],$$

where $\varsigma = \inf\{t > 0 : \Theta_t = \dagger\}$. 

Roughly speaking, one thinks of a MAP as a ‘Markov modulated’ Lévy process. It has ‘conditional stationary and independent increments’. Think of the $E$-valued Markov process $\Theta$ as modulating the characteristics of $\xi$ (which would otherwise be a Lévy process).
Lamperti–Kiu transform

In order to introduce the analogue of the Lamperti transform in $d$-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

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- Roughly speaking, one thinks of a MAP as a ‘Markov modulated’ Lévy process
- It has ‘conditional stationary and independent increments’
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In order to introduce the analogue of the Lamperti transform in $d$-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

**Definition**

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $P_{x,\theta}, x \in \mathbb{R}, \theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a Markov additive process (MAP) if $\Theta$ is a regular Feller process on $E$ with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \rightarrow \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on \{t < \varsigma\},

$$E_{x,\theta}[f(\xi_{t+s} - \xi_t, \Theta_{t+s}) | \sigma((\xi_u, \Theta_u), u \leq t)] = E_{0,\Theta_t}[f(\xi_s, \Theta_s)],$$

where $\varsigma = \inf\{t > 0 : \Theta_t = \dagger\}$.

- Roughly speaking, one thinks of a MAP as a ‘Markov modulated’ Lévy process
- It has ‘conditional stationary and independent increments’
- Think of the $E$-valued Markov process $\Theta$ as modulating the characteristics of $\xi$ (which would otherwise be a Lévy processes).
Lamperti–Kiu transform

Theorem
Fix $\alpha > 0$. The process $Z$ is a ssMp with index $\alpha$ if and only if there exists a (killed) MAP, $(\xi, \Theta)$ on $\mathbb{R} \times S_{d-1}$ such that

$$Z_t := e^{\xi \varphi(t)} \Theta \varphi(t), \quad t \leq I_\zeta,$$

where

$$\varphi(t) = \inf \left\{ s > 0 : \int_0^s e^{\alpha u} \, du > t \right\}, \quad t \leq I_\zeta,$$

and $I_\zeta = \int_0^\zeta e^{\alpha s} \, ds$ is the lifetime of $Z$ until absorption at the origin. Here, we interpret $\exp\{-\infty\} \times \uparrow := 0$ and $\inf \emptyset := \infty$.

$\blacktriangleright$ In the above representation, the time to absorption in the origin,

$$\zeta = \inf \{ t > 0 : Z_t = 0 \},$$

satisfies $\zeta = I_\zeta$.

$\blacktriangleright$ Note $x \in \mathbb{R}^d$ if and only if

$$x = (|x|, \text{Arg}(x)),$$

where $\text{Arg}(x) = x/|x| \in S_{d-1}$. The Lamperti–Kiu decomposition therefore gives us a $d$-dimensional skew product decomposition of self-similar Markov processes.
The stable process $X$ is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above.
LAMPERTI-STABLE MAP

- The stable process $X$ is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above.
- How do we characterise its underlying MAP $(\xi, \Theta)$?
Lamperti-stable MAP

- The stable process $X$ is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above.
- **How do we characterise its underlying MAP $(\xi, \Theta)$?**
- We already know that $|X|$ is a positive similar Markov process and hence $\xi$ is a Lévy process, albeit correlated to $\Theta$. 
The stable process $X$ is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above.

**How do we characterise its underlying MAP $(\xi, \Theta)$?**

We already know that $|X|$ is a positive similar Markov process and hence $\xi$ is a Lévy process, albeit corollated to $\Theta$.

What properties does $\Theta$ and what properties to the pair $(\xi, \Theta)$ have?
MAP ISOTROPY

Theorem
Suppose $(\xi, \Theta)$ is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), P_{x,\theta})$ is equal in law to $((\xi, \Theta), P_{\log|x|, \theta})$, for every orthogonal $d$-dimensional matrix $U$ and $x \in \mathbb{R}^d, \theta \in S_{d-1}$. 
Theorem
Suppose \((\xi, \Theta)\) is the MAP underlying the stable process. Then \(((\xi, U^{-1}\Theta), P_{x,\theta})\) is equal in law to \(((\xi, \Theta), P_{x,U^{-1}\theta})\), for every orthogonal \(d\)-dimensional matrix \(U\) and \(x \in \mathbb{R}^d\), \(\theta \in \mathbb{S}_{d-1}\).

Proof.
First note that \(\varphi(t) = \int_0^t |X_u|^{-\alpha} du\). It follows that

\[
(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \text{Arg}(X_{A(t)})), \quad t \geq 0,
\]

where the random times \(A(t) = \inf \{ s > 0 : \int_0^s |X_u|^{-\alpha} du > t \}\) are stopping times in the natural filtration of \(X\).
MAP ISOTROPY

Theorem
Suppose \((\xi, \Theta)\) is the MAP underlying the stable process. Then \(((\xi, U^{-1} \Theta), P_{x, \theta})\) is equal in law to \(((\xi, \Theta), P_{x, U^{-1} \Theta})\), for every orthogonal \(d\)-dimensional matrix \(U\) and \(x \in \mathbb{R}^d, \theta \in S_{d-1}\).

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First note that \(\varphi(t) = \int_0^t |X_u|^{-\alpha} \, du\). It follows that
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(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \text{Arg}(X_{A(t)})), \quad t \geq 0,
\]
where the random times \(A(t) = \inf \{ s > 0 : \int_0^s |X_u|^{-\alpha} \, du > t \}\) are stopping times in the natural filtration of \(X\).

Now suppose that \(U\) is any orthogonal \(d\)-dimensional matrix and let \(X' = U^{-1}X\). Since \(X\) is isotropic and since \(|X'| = |X|\), and \(\text{Arg}(X') = U^{-1}\text{Arg}(X)\), we see that, for \(x \in \mathbb{R}\) and \(\theta \in S_{d-1}\)
\[
((\xi, U^{-1} \Theta), P_{\log |x|, \theta}) = ((\log |X_{A(\cdot)}|, U^{-1}\text{Arg}(X_{A(\cdot)})), P_x)
\]
\[
\overset{d}{=} ((\log |X_{A(\cdot)}|, \text{Arg}(X_{A(\cdot)})), P_{U^{-1}x})
\]
\[
= ((\xi, \Theta), P_{\log |x|, U^{-1} \theta})
\]
as required.
MAP CORRELATION

- We will work with the increments \( \Delta \xi_t = \xi_t - \xi_{t-} \in \mathbb{R}, t \geq 0 \),
We will work with the increments $\Delta \xi_t = \xi_t - \xi_{t-} \in \mathbb{R}$, $t \geq 0$.

**Theorem (Bo Li, Victor Rivero, Bertoin-Werner (1996))**

Suppose that $f$ is a bounded measurable function on $[0, \infty) \times \mathbb{R} \times \mathbb{R} \times S_{d-1} \times S_{d-1}$ such that $f(\cdot, \cdot, 0, \cdot, \cdot) = 0$, then, for all $\theta \in S_{d-1}$,

$$E_{0,\theta} \left( \sum_{s>0} f(s, \xi_{s-}, \Delta \xi_s, \Theta_{s-}, \Theta_s) \right)$$

$$= \int_0^\infty \int_\mathbb{R} \int_{S_{d-1}} \int_{S_{d-1}} \int_\mathbb{R} V_\theta(ds, dx, d\vartheta) \sigma_1(d\phi)dy \frac{c(\alpha)e^{yd}}{|e^y \phi - \vartheta|^{\alpha+d}} f(s, x, y, \vartheta, \phi),$$

where

$$V_\theta(ds, dx, d\vartheta) = P_{0,\theta}(\xi_s \in dx, \Theta_s \in d\vartheta)ds,$$

is the space-time potential of $(\xi, \Theta)$ under $P_{0,\theta}$, $\sigma_1(\phi)$ is the surface measure on $S_{d-1}$ normalised to have unit mass and

$$c(\alpha) = 2^{\alpha-1} \pi^{-d} \Gamma((d + \alpha)/2) \Gamma(d/2)/|\Gamma(-\alpha/2)|.$$

MAP OF \((X, \mathbb{P}^o)\)

- Recall that \(|X_t|^{\alpha-d}, t \geq 0\), is a martingale.
- Informally, we should expect \(\mathcal{L}h = 0\), where \(h(x) = |x|^{\alpha-d}\) and \(\mathcal{L}\) is the infinitesimal generator of the stable process, which has action

\[
\mathcal{L}f(x) = a \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x + y) - f(x) - 1_{(|y| \leq 1)} y \cdot \nabla f(x)] \Pi(dy), \quad |x| > 0,
\]
for appropriately smooth functions.
MAP OF \((X, \mathbb{P}_0)\)

- Recall that \((|X_t|^{\alpha-d}, t \geq 0)\), is a martingale.
- Informally, we should expect \(Lh = 0\), where \(h(x) = |x|^{\alpha-d}\) and \(L\) is the infinitesimal generator of the stable process, which has action

\[
Lf(x) = a \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x + y) - f(x) - 1_{(|y| \leq 1)}y \cdot \nabla f(x)] \Pi(\text{d}y), \quad |x| > 0,
\]

for appropriately smooth functions.
- Associated to \((X, \mathbb{P}_x), x \neq 0\) is the generator

\[
L^\circ f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x^\circ [f(X_t)] - f(x)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_x[|X_t|^{\alpha-d}f(X_t)] - |x|^{\alpha-d}f(x)}{|x|^{\alpha-d}t},
\]
MAP OF \((X, \mathbb{P}^\circ)\)

- Recall that \((|X_t|^{\alpha-d}, t \geq 0)\), is a martingale.
- Informally, we should expect \(\mathcal{L}h = 0\), where \(h(x) = |x|^{\alpha-d}\) and \(\mathcal{L}\) is the infinitesimal generator of the stable process, which has action

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\mathcal{L}f(x) = a \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x + y) - f(x) - 1_{|y| \leq 1}y \cdot \nabla f(x)] \Pi(dy), \quad |x| > 0,
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- Associated to \((X, \mathbb{P}_x), x \neq 0\) is the generator

\[
\mathcal{L}^\circ f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x^\circ f(X_t) - f(x)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_x [\alpha^{\alpha-d}f(X_t) - |x|^{\alpha-d}f(x)]}{|x|^{\alpha-d}t},
\]

- That is to say

\[
\mathcal{L}^\circ f(x) = \frac{1}{h(x)} \mathcal{L}(hf)(x),
\]
MAP OF \((X, \mathbb{P}_0)\)

- Recall that \((|X_t|^\alpha - d, t \geq 0)\), is a martingale.
- Informally, we should expect \(Lh = 0\), where \(h(x) = |x|^{\alpha - d}\) and \(L\) is the infinitesimal generator of the stable process, which has action

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Lf(x) = a \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x + y) - f(x) - 1_{(|y| \leq 1)} y \cdot \nabla f(x)] \Pi(dy), \quad |x| > 0,
\]

for appropriately smooth functions.
- Associated to \((X, \mathbb{P}_x), x \neq 0\) is the generator

\[
L^\circ f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x^0[f(X_t)] - f(x)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_x[|X_t|^{\alpha - d}f(X_t)] - |x|^{\alpha - d}f(x)}{|x|^{\alpha - d}t},
\]

- That is to say

\[
L^\circ f(x) = \frac{1}{h(x)} L(hf)(x),
\]

- Straightforward algebra using \(Lh = 0\) gives us

\[
L^\circ f(x) = a \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x + y) - f(x) - 1_{(|y| \leq 1)} y \cdot \nabla f(x)] \frac{h(x + y)}{h(x)} \Pi(dy), \quad |x| > 0
\]
MAP OF \((X, \mathbb{P}^\circ)\)

- Recall that \(|X_t|^{\alpha-d}, t \geq 0\), is a martingale.
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L^\circ f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x^\circ [f(X_t)] - f(x)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_x [|X_t|^{\alpha-d}f(X_t)] - |x|^{\alpha-d}f(x)}{|x|^{\alpha-d}t},
\]

- That is to say

\[
L^\circ f(x) = \frac{1}{h(x)} L(hf)(x),
\]

- Straightforward algebra using \(Lh = 0\) gives us

\[
L^\circ f(x) = a \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y| \leq 1)} y \cdot \nabla f(x)] \frac{h(x+y)}{h(x)} \Pi(dy), \quad |x| > 0
\]

- Equivalently, the rate at which \((X, \mathbb{P}^\circ_x), x \neq 0\) jumps given by

\[
\Pi^\circ(x, B) := \frac{2^{\alpha-1} \Gamma((d + \alpha)/2) \Gamma(d/2)}{\pi^d |\Gamma(-\alpha/2)|} \int_{S_{d-1}} d\sigma_1(\phi) \int_{(0, \infty)} \mathbf{1}_B(r\phi) \frac{dr}{r^{\alpha+1}} \frac{|x + r\phi|^{\alpha-d}}{|x|^{\alpha-d}},
\]

for \(|x| > 0\) and \(B \in \mathcal{B}(\mathbb{R}^d)\).
MAP of \((X, \mathbb{P}^0)\)

**Theorem**

Suppose that \(f\) is a bounded measurable function on \([0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}\) such that \(f(\cdot, \cdot, 0, \cdot, \cdot) = 0\), then, for all \(\theta \in \mathbb{S}_{d-1}\),

\[
\mathbb{E}_{0, \theta}^0 \left( \sum_{s > 0} f(s, \xi_s-, \Delta \xi_s, \Theta_s-, \Theta_s) \right) = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V_{\theta}^o(ds, dx, d\vartheta) \sigma_1(d\phi) dy \frac{c(\alpha)e^{yd}}{|e^y\phi - \vartheta|^{\alpha+d}} f(s, x, -y, \vartheta, \phi),
\]

where

\[
V_{\theta}^o(ds, dx, d\vartheta) = \mathbb{P}_{0, \theta}^o(\xi_s \in dx, \Theta_s \in d\vartheta)ds, \quad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,
\]

is the space-time potential of \((\xi, \Theta)\) under \(\mathbb{P}_{0, \theta}^o\).

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of \((\xi, \Theta)\) under \(\mathbb{P}_{0, \theta}^o, x \in \mathbb{R}, \theta \in \mathbb{S}_{d-1}\), is precisely that of \((-\xi, \Theta)\) under \(\mathbb{P}_{x, \theta}, x \in \mathbb{R}, \theta \in \mathbb{S}_{d-1}\).
Theorem

Suppose that $f$ is a bounded measurable function on $[0, \infty) \times \mathbb{R} \times \mathbb{R} \times S_{d-1} \times S_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in S_{d-1},$

$$E_{0,\theta}\left(\sum_{s>0} f(s, \xi_s-, \Delta \xi_s, \Theta_s-, \Theta_s)\right)
= \int_0^\infty \int_{\mathbb{R}} \int_{S_{d-1}} \int_{S_{d-1}} \int_{\mathbb{R}} V_\theta(ds, dx, d\vartheta)\sigma_1(d\varphi)dy \frac{c(\alpha)e^{yd}}{|e^y \phi - \vartheta|^{\alpha+d}}f(s, x, y, \vartheta, \phi),$$

where

$$V_\theta(ds, dx, d\vartheta) = P_{0,\theta}(\xi_s \in dx, \Theta_s \in d\vartheta)ds, \quad x \in \mathbb{R}, \vartheta \in S_{d-1}, s \geq 0,$$

is the space-time potential of $(\xi, \Theta)$ under $P_{0,\theta}^\circ$.

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of $(\xi, \Theta)$ under $P_{0,\theta}^\circ, x \in \mathbb{R}, \theta \in S_{d-1}$, is precisely that of $(-\xi, \Theta)$ under $P_{x,\theta}, x \in \mathbb{R}, \theta \in S_{d-1}$. 
§9. Riesz–Bogdan–Žak transform
Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

$$Kx = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$
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$$Kx = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$ 

This transformation inverts space through the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$. 

Theorem (d-dimensional Riesz–Bogdan–˙Zak Transform, $d \geq 2$)

Suppose that $X$ is a $d$-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf \{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}, \quad t \geq 0.$$ 

(1)

Then, for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$(KX, t \geq 0) \text{ under } P_x \text{ is equal in law to } (X, P^x \circ Kx).$$
Riesz–Bogdan–Żak Transform

Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

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This transformation inverts space through the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$.

Write $x \in \mathbb{R}^d$ in skew product form $x = (|x|, \text{Arg}(x))$, and note that

$$Kx = (|x|^{-1}, \text{Arg}(x)), \quad x \in \mathbb{R}^d \setminus \{0\},$$

showing that the $K$-transform ‘radially inverts’ elements of $\mathbb{R}^d$ through $S_{d-1}$. 

Theorem ($d$-dimensional Riesz–Bogdan–Żak Transform, $d \geq 2$)

Suppose that $X$ is a $d$-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^\alpha d u > t\}, \quad t \geq 0.$$ (1)

Then, for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$(KX_{\eta(t)}, t \geq 0)$$

under $P_x$ is equal in law to $(X, P_x \circ Kx)$. 

**Riesz–Bogdan–Žak Transform**

- Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by
  \[
  Kx = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}.
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  showing that the $K$-transform ‘radially inverts’ elements of $\mathbb{R}^d$ through $S_{d-1}$.
- In particular $K(Kx) = x$
Define the transformation \( K : \mathbb{R}^d \mapsto \mathbb{R}^d \), by

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Kx = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}.
\]

This transformation inverts space through the unit sphere \( \{x \in \mathbb{R}^d : |x| = 1\} \).

Write \( x \in \mathbb{R}^d \) in skew product form \( x = (|x|, \text{Arg}(x)) \), and note that

\[
Kx = (|x|^{-1}, \text{Arg}(x)), \quad x \in \mathbb{R}^d \setminus \{0\},
\]

showing that the \( K \)-transform ‘radially inverts’ elements of \( \mathbb{R}^d \) through \( S_{d-1} \).

In particular \( K(Kx) = x \)

---

**Theorem (d-dimensional Riesz–Bogdan–Žak Transform, \( d \geq 2 \))**

Suppose that \( X \) is a \( d \)-dimensional isotropic stable process with \( d \geq 2 \). Define

\[
\eta(t) = \inf \{ s > 0 : \int_0^s |X_u|^{-2\alpha} \, du > t \}, \quad t \geq 0.
\]

Then, for all \( x \in \mathbb{R}^d \setminus \{0\}, (KX_{\eta(t)}, t \geq 0) \) under \( \mathbb{P}_x \) is equal in law to \( (X, \mathbb{P}_{Kx}^\circ) \).
Proof of Riesz–Bogdan–˙Zak Transform

We give a proof, different to the original proof of Bogdan and ˙Zak (2010).

- Recall that $X_t = e^{\xi \varphi(t)} \Theta_{\varphi(t)}$, where

$$\int_0^{\varphi(t)} e^{\alpha \xi u} \, du = t, \quad t \geq 0.$$
**Proof of Riesz–Bogdan–Žak Transform**

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  \[
  \int_0 \varphi(t) e^{\alpha \xi u} \, du = t, \quad t \geq 0.
  \]

- Note also that, as an inverse,
  \[
  \int_0 \eta(t) |X_u|^{-2\alpha} \, du = t, \quad t \geq 0.
  \]
PROOF OF RIESZ–BOGDAN–ŽAK TRANSFORM

We give a proof, different to the original proof of Bogdan and Žak (2010).

Recall that $X_t = e^{\xi \varphi(t)} \Theta \varphi(t)$, where

$$\int_0^{\varphi(t)} e^{\alpha \xi u} \, du = t, \quad t \geq 0.$$  

Note also that, as an inverse,

$$\int_0^{\eta(t)} |X_u|^{-2\alpha} \, du = t, \quad t \geq 0.$$  

Differentiating,

$$\frac{d\varphi(t)}{dt} = e^{-\alpha \xi \varphi(t)} \text{ and } \frac{d\eta(t)}{dt} = e^{2\alpha \xi \varphi \circ \eta(t)}, \quad \eta(t) < \tau\{0\}.$$  

and chain rule now tells us that

$$\frac{d(\varphi \circ \eta)(t)}{dt} = \left. \frac{d\varphi(s)}{ds} \right|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha \xi \varphi \circ \eta(t)}.$$
PROOF OF RIESZ–BOGDAN–ŽAK TRANSFORM

We give a proof, different to the original proof of Bogdan and Žak (2010).

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$$\frac{d\varphi(t)}{dt} = e^{-\alpha \xi \varphi(t)} \quad \text{and} \quad \frac{d\eta(t)}{dt} = e^{2\alpha \xi \varphi \circ \eta(t)}, \quad \eta(t) < \tau\{0\}.$$  

and chain rule now tells us that

$$\frac{d(\varphi \circ \eta)(t)}{dt} = \frac{d\varphi(s)}{ds} \bigg|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha \xi \varphi \circ \eta(t)}.$$  

- Said another way,

$$\int_0^{\varphi \circ \eta(t)} e^{-\alpha \xi u} \, du = t, \quad t \geq 0,$$

or

$$\varphi \circ \eta(t) = \inf\{s > 0 : \int_0^s e^{-\alpha \xi u} \, du > t\}.$$
Next note that

\[ KX_{\eta(t)} = e^{-\xi \varphi \circ \eta(t)} \Theta_{\varphi \circ \eta(t)}, \quad t \geq 0, \]

and we have just shown that

\[ \varphi \circ \eta(t) = \inf\{s > 0 : \int_0^s e^{-\alpha \xi_u} du > t\}. \]
Proof of Riesz–Bogdan–Žak Transform

Next note that
\[ KX_{\eta}(t) = e^{-\xi \varphi \circ \eta(t)} \Theta \varphi \circ \eta(t), \quad t \geq 0, \]
and we have just shown that
\[ \varphi \circ \eta(t) = \inf \{ s > 0 : \int_{0}^{s} e^{-\alpha \xi u} du > t \}. \]

It follows that \((KX_{\eta}(t), t \geq 0)\) is a self-similar Markov process with underlying MAP \((-\xi, \Theta)\).
Next note that

\[ KX_\eta(t) = e^{-\xi \varphi \circ \eta(t)} \Theta \varphi \circ \eta(t), \quad t \geq 0, \]

and we have just shown that

\[ \varphi \circ \eta(t) = \inf \{ s > 0 : \int_0^s e^{-\alpha \xi u} du > t \}. \]

It follows that \((KX_\eta(t), t \geq 0)\) is a self-similar Markov process with underlying MAP \((-\xi, \Theta)\).

We have also seen that \((X, \mathbb{P}^x), x \neq 0,\) is also a self-similar Markov process with underlying MAP given by \((-\xi, \Theta)\).
Proof of Riesz–Bogdan–Žak transform

Next note that

\[ KX_\eta(t) = e^{-\xi \varphi \circ \eta(t)} \Theta \varphi \circ \eta(t), \quad t \geq 0, \]

and we have just shown that

\[ \varphi \circ \eta(t) = \inf\{s > 0 : \int_0^s e^{-\alpha \xi u} \, du > t\}. \]

It follows that \((KX_\eta(t), t \geq 0)\) is a self-similar Markov process with underlying MAP \((-\xi, \Theta)\).

We have also seen that \((X, \mathbb{P}^\circ_x), x \neq 0\), is also a self-similar Markov process with underlying MAP given by \((-\xi, \Theta)\).

The statement of the theorem follows.
§10. Hitting spheres
PORT’S SPHERE HITTING PROBABILITY

- Recall that a stable process cannot hit points
PORT’S SPHERE HITTING PROBABILITY

- Recall that a stable process cannot hit points
- We are ultimately interested in the distribution of the position of $X$ on first hitting of the sphere $S_{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}$. 

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- Define
  $$\tau^\circ = \inf\{ t > 0 : |X_t| = 1 \}.$$
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We are ultimately interested in the distribution of the position of $X$ on first hitting of the sphere $S_{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}$.
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We start with an easier result
Port’s Sphere hitting probability

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  $$\tau^\circ = \inf\{t > 0 : |X_t| = 1\}.$$

- We start with an easier result

**Theorem (Port (1969))**

*If $\alpha \in (1, 2)$, then*

$$\mathbb{P}_x(\tau^\circ < \infty) = \frac{\Gamma \left( \frac{\alpha + d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma(\alpha - 1)} \left\{ \begin{array}{ll} 2F_1((d - \alpha)/2, 1 - \alpha/2, d/2; |x|^2) & 1 > |x| \\ |x|^{\alpha-d}2F_1((d - \alpha)/2, 1 - \alpha/2, d/2; 1/|x|^2) & 1 \leq |x|. \end{array} \right.$$  

*Otherwise, if $\alpha \in (0, 1]$, then $\mathbb{P}_x(\tau^\circ = \infty) = 1$ for all $x \in \mathbb{R}^d$.***
Proof of Port’s hitting probability

- If \((\xi, \Theta)\) is the underlying MAP then

\[
P_x(\tau^{\odot} < \infty) = P_{\log |x|}(\tau^{\{0\}} < \infty) = P_0(\tau^{\{\log(1/|x|)\}} < \infty),
\]

where \(\tau^{\{z\}} = \inf\{t > 0 : \xi_t = z\}, z \in \mathbb{R}\). (Note, the time change in the Lamperti–Kiu representation does not level out.)
PROOF OF PORT’S HITTING PROBABILITY

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Using Sterling’s formula, we have, \(|\Gamma(x + iy)| = \sqrt{2\pi}e^{-\frac{\pi}{2}|y|}|y|^{x-\frac{1}{2}}(1 + o(1))\), for \(x, y \in \mathbb{R}\), as \(y \to \infty\), uniformly in any finite interval \(-\infty < a \leq x \leq b < \infty\).

Hence,

\[
\frac{1}{\Psi(z)} = \frac{\Gamma(-\frac{1}{2}iz)}{\Gamma(\frac{1}{2}(-iz + \alpha))} \frac{\Gamma(\frac{1}{2}(iz + d - \alpha))}{\Gamma(\frac{1}{2}(iz + d))} \sim |z|^{-\alpha}
\]

uniformly on \(\mathbb{R}\) as \(|z| \to \infty\).
### Proof of Port’s Hitting Probability

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  \]

  where \(\tau^z = \inf\{t > 0 : \xi_t = z\}, z \in \mathbb{R}\). (Note, the time change in the Lamperti–Kiu representation does not level out.)

- Using Sterling’s formula, we have, \(|\Gamma(x + iy)| = \sqrt{2\pi} e^{-\frac{\pi}{2} |y|} |y|^{x-\frac{1}{2}} (1 + o(1))\), for \(x, y \in \mathbb{R}\), as \(y \to \infty\), uniformly in any finite interval \(-\infty < a \leq x \leq b < \infty\). Hence,
  
  \[
  \frac{1}{\Psi(z)} = \frac{\Gamma(-\frac{1}{2}iz)}{\Gamma(\frac{1}{2}(-iz + \alpha))} \frac{\Gamma(\frac{1}{2}(iz + d - \alpha))}{\Gamma(\frac{1}{2}(iz + d))} \sim |z|^{-\alpha}
  \]

  uniformly on \(\mathbb{R}\) as \(|z| \to \infty\).

- From Kesten-Brestagnolle integral test we conclude that \((1 + \Psi(z))^{-1}\) is integrable and each sphere \(S_{d-1}\) can be reached with positive probability from any \(x\) with \(|x| \neq 1\) if and only if \(\alpha \in (1, 2)\).
**Proof of Port’s Hitting Probability**

- Note that

\[
\frac{\Gamma\left(\frac{1}{2}(-iz + \alpha)\right)}{\Gamma\left(-\frac{1}{2}iz\right)} \frac{\Gamma\left(\frac{1}{2}(iz + d)\right)}{\Gamma\left(\frac{1}{2}(iz + d - \alpha)\right)}
\]

so that \(\Psi(-iz)\), is well defined for \(\text{Re}(z) \in (-d, \alpha)\) with roots at 0 and \(\alpha - d\).
**Proof of Port’s hitting probability**

- Note that

\[
\frac{\Gamma\left(\frac{1}{2}(-iz + \alpha)\right)}{\Gamma(-\frac{1}{2}iz)} \cdot \frac{\Gamma\left(\frac{1}{2}(iz + d)\right)}{\Gamma\left(\frac{1}{2}(iz + d - \alpha)\right)}
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so that \(\Psi(-iz)\), is well defined for \(\text{Re}(z) \in (-d, \alpha)\) with roots at 0 and \(\alpha - d\).

- We can use the identity

\[
\mathbb{P}_x(\tau^\odot < \infty) = \frac{u_\xi(\log(1/|x|))}{u_\xi(0)},
\]

providing

\[
u_\xi(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \frac{e^{-zx}}{\Psi(-iz)} \, dz, \quad x \in \mathbb{R},
\]

for \(c \in (\alpha - d, 0)\).
**Proof of Port's hitting probability**

- Note that
  \[
  \frac{\Gamma\left(\frac{1}{2}(-iz + \alpha)\right)}{\Gamma\left(-\frac{1}{2}iz\right)} \frac{\Gamma\left(\frac{1}{2}(iz + d)\right)}{\Gamma\left(\frac{1}{2}(iz + d - \alpha)\right)}
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  providing
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  u_\xi(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \frac{e^{-zx}}{\Psi(-iz)} \, dz,
  \]
  for \(c \in (\alpha - d, 0)\).

- Build the contour integral around simple poles at \(\{-2n - (d - \alpha) : n \geq 0\}\).

\[
\frac{1}{2\pi i} \int_{c-i\mathbb{R}}^{c+i\mathbb{R}} \frac{e^{-zx}}{\Psi(-iz)} \, dz
= -\frac{1}{2\pi i} \int_{c+Re^{i\theta} : \theta \in \left(\pi/2, 3\pi/2\right)} \frac{e^{-zx}}{\Psi(-iz)} \, dz
+ \sum_{1 \leq n \leq [R]} \text{Res}\left(\frac{e^{-zx}}{\Psi(-iz)} ; z = -2n - (d - \alpha)\right).
\]
**Proof of Port’s hitting probability**

- Now fix $x \leq 0$ and recall estimate $|1/\Psi(-iz)| \lesssim |z|^{-\alpha}$. The assumption $x \leq 0$ and the fact that the arc length of $\{c + Re^{i\theta} : \theta \in (\pi/2, 3\pi/2)\}$ is $\pi R$, gives us

  $$\left| \int_{c + Re^{i\theta} : \theta \in (\pi/2, 3\pi/2)} \frac{e^{-xz}}{\Psi(-iz)} \, dz \right| \leq CR^{-(\alpha-1)} \to 0$$

as $R \to \infty$ for some constant $C > 0$. 
PROOF OF PORT’S HITTING PROBABILITY

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as $R \to \infty$ for some constant $C > 0$.

Moreover,

$$u_\xi(x) = \sum_{n \geq 1} \text{Res} \left( \frac{e^{-zx}}{\Psi(-iz)}; z = -2n - (d - \alpha) \right)$$

$$= \sum_{0}^{\infty} (-1)^{n+1} \frac{\Gamma(n + (d - \alpha)/2)}{\Gamma(-n + \alpha/2)\Gamma(n + d/2)} \frac{e^{2nx}}{n!}$$

$$= e^{x(d-\alpha)} \frac{\Gamma((d - \alpha)/2)}{\Gamma(d/2)\Gamma(\alpha/2)} _2F_1((d - \alpha)/2, 1 - \alpha/2, d/2; e^{2x}),$$

Which also gives a value for $u_\xi(0)$. 
PROOF OF PORT’S HITTING PROBABILITY

Now fix $x \leq 0$ and recall estimate $|1/\Psi(-iz)| \lesssim |z|^{-\alpha}$. The assumption $x \leq 0$ and the fact that the arc length of $\{c + Re^{i\theta} : \theta \in (\pi/2, 3\pi/2)\}$ is $\pi R$, gives us

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Moreover,

$$u_\xi(x) = \sum_{n \geq 1} \text{Res} \left( \frac{e^{-zx}}{\Psi(-iz)} ; z = -2n - (d - \alpha) \right)$$

$$= \sum_{n \geq 0} (-1)^{n+1} \frac{\Gamma(n + (d - \alpha)/2)}{\Gamma(-n + \alpha/2)\Gamma(n + d/2)} \frac{e^{2nx}}{n!}$$

$$= e^{x(d - \alpha)} \frac{\Gamma((d - \alpha)/2)}{\Gamma(\alpha/2)\Gamma(d/2)} 2F_1((d - \alpha)/2, 1 - \alpha/2, d/2; e^{2x}),$$

Which also gives a value for $u_\xi(0)$.

Hence, for $1 \leq |x|$,  

$$\mathbb{P}_x(\tau^\circ \lt \infty) = \frac{u_\xi(\log(1/|x|))}{u_\xi(0)}$$

$$= \frac{\Gamma \left( \frac{\alpha + d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma(\alpha - 1)} |x|^{\alpha - d} 2F_1((d - \alpha)/2, 1 - \alpha/2, d/2; |x|^{-2}).$$
Proof of Port’s hitting probability

- To deal with the case $|x| < 1$, we can appeal to the Riesz–Bogdan–Żak transform to help us.
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To this end we note that, for $|x| < 1$, $|Kx| > 1$

$$P_{Kx}(\tau^{\circ} < \infty) = P_{x}(\tau^{\circ} < \infty) = E_{x} \left[ \frac{|X_{\tau^{\circ}}|^{\alpha-d}}{|x|^{\alpha-d}} 1_{(\tau^{\circ} < \infty)} \right] = \frac{1}{|x|^{\alpha-d}} P_{x}(\tau^{\circ} < \infty)$$
Proof of Port’s hitting probability

- To deal with the case $|x| < 1$, we can appeal to the Riesz–Bogdan–Żak transform to help us.
- To this end we note that, for $|x| < 1$, $|Kx| > 1$

$$\mathbb{P}_{Kx}(\tau^\odot < \infty) = \mathbb{P}_x^{\odot}(\tau^\odot < \infty) = \mathbb{E}_x \left[ \frac{|X_{\tau^\odot}|^{\alpha-d}}{|x|^{\alpha-d}} 1_{(\tau^\odot < \infty)} \right] = \frac{1}{|x|^{\alpha-d}} \mathbb{P}_x(\tau^\odot < \infty)$$

- Hence plugging in the expression for $|x| < 1$,

$$\mathbb{P}_x(\tau^\odot < \infty) = \frac{\Gamma \left( \frac{\alpha+d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma(\alpha - 1)} {}_2F_1((d - \alpha)/2, 1 - \alpha/2, d/2; |x|^2),$$

thus completing the proof.

- To deal with the case $x = 0$, take limits in the established identity as $|x| \to 0$. 
Riesz representation of Port’s hitting probability

Theorem
Suppose $\alpha \in (1, 2)$. For all $x \in \mathbb{R}^d$,

$$\mathbb{P}_x(\tau^< < \infty) = \frac{\Gamma \left( \frac{\alpha + d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma (\alpha - 1)} \int_{S_{d-1}} |z - x|^{\alpha - d} \sigma_1(dz),$$

where $\sigma_1(dz)$ is the uniform measure on $S_{d-1}$, normalised to have unit mass. In particular, for $y \in S_{d-1}$,

$$\int_{S_{d-1}} |z - y|^{\alpha - d} \sigma_1(dz) = \frac{\Gamma \left( \frac{d}{2} \right) \Gamma (\alpha - 1)}{\Gamma \left( \frac{\alpha + d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)}.$$
\textbf{Proof of Riesz representation of Port's hitting probability}

- We know that $|X_t - z|^{\alpha-d}, t \geq 0$ is a martingale.
PROOF OF RIESZ REPRESENTATION OF PORT'S HITTING PROBABILITY

- We know that $|X_t - z|^{\alpha - d}, t \geq 0$ is a martingale.
- Hence we know that

$$M_t := \int_{S_{d-1}} |z - X_{t \wedge \tau \circ}|^{\alpha - d} \sigma_1(dz), \quad t \geq 0,$$

is a martingale.
Proof of Riesz representation of Port's hitting probability

- We know that $|X_t - z|^\alpha - d, t \geq 0$ is a martingale.
- Hence we know that
  \[ M_t := \int_{S_{d-1}} |z - X_t \wedge \tau \circ|^{\alpha - d} \sigma_1(dz), \quad t \geq 0, \]
  is a martingale.
- Recall that $\lim_{t \to \infty} |X_t| = 0$ and $\alpha < d$ and hence
  \[ M_\infty := \lim_{t \to \infty} M_t = \int_{S_{d-1}} |z - X_\tau \circ|^{\alpha - d} \sigma_1(dz)1(\tau \circ < \infty) = C1(\tau \circ < \infty). \]
  where, despite the randomness in $X_\tau \circ$, by rotational symmetry,
  \[ C = \int_{S_{d-1}} |z - 1|^{\alpha - d} \sigma_1(dz), \]
  and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the 'North Pole' on $S_{d-1}$. 
PROOF OF RIESZ REPRESENTATION OF PORT’S HITTING PROBABILITY

- We know that $|X_t - z|^\alpha - d, \ t \geq 0$ is a martingale.
- Hence we know that
  $$M_t := \int_{S_{d-1}} |z - X_{t\land \tau \circ}|^{\alpha - d} \sigma_1(dz), \quad t \geq 0,$$
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- Recall that $\lim_{t \to \infty} |X_t| = 0$ and $\alpha < d$ and hence
  $$M_\infty := \lim_{t \to \infty} M_t = \int_{S_{d-1}} |z - X_{\tau \circ}|^{\alpha - d} \sigma_1(dz) 1_{(\tau \circ < \infty)} \overset{d}{=} C 1_{(\tau \circ < \infty)}.$$

  where, despite the randomness in $X_{\tau \circ}$, by rotational symmetry,
  $$C = \int_{S_{d-1}} |z - 1|^{\alpha - d} \sigma_1(dz),$$

  and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the ‘North Pole’ on $S_{d-1}$.
- Since $M$ is a UI martingale, taking expectations of $M_\infty$
  $$\int_{S_{d-1}} |z - x|^{\alpha - d} \sigma_1(dz) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] = \mathbb{C}P_x(\tau \circ < \infty)$$
PROOF OF RIESZ REPRESENTATION OF PORT’S HITTING PROBABILITY

▷ We know that $|X_t - z|^{\alpha - d}, t \geq 0$ is a martingale.

▷ Hence we know that

$$M_t := \int_{S_{d-1}} |z - X_{t \wedge \tau}^\circ|^{\alpha - d} \sigma_1(dz), \quad t \geq 0,$$

is a martingale.

▷ Recall that $\lim_{t \to \infty} |X_t| = 0$ and $\alpha < d$ and hence

$$M_\infty := \lim_{t \to \infty} M_t = \int_{S_{d-1}} |z - X_{\tau}^\circ|^{\alpha - d} \sigma_1(dz) \mathbf{1}_{(\tau^\circ < \infty)} = \frac{d}{\Gamma(\alpha - 1)} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\alpha - \frac{d}{2}ight),$$

where, despite the randomness in $X_{\tau^\circ}$, by rotational symmetry,

$$C = \int_{S_{d-1}} |z - 1|^{\alpha - d} \sigma_1(dz),$$

and $1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ is the ‘North Pole’ on $S_{d-1}$.

▷ Since $M$ is a UI martingale, taking expectations of $M_\infty$

$$\int_{S_{d-1}} |z - x|^{\alpha - d} \sigma_1(dz) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty] = \mathbb{C} \mathbb{P}_x(\tau^\circ < \infty)$$

▷ Taking limits as $|x| \to 0$,

$$C = 1/\mathbb{P}(\tau^\circ < \infty) = \Gamma\left(\frac{d}{2}\right) \Gamma(\alpha - 1)/\Gamma\left(\alpha - \frac{d}{2}ight) \Gamma\left(\frac{\alpha + d}{2} - 1\right) \Gamma\left(\frac{\alpha}{2}\right).$$
Sphere inversions
Sphere Inversions

- Fix a point \( b \in \mathbb{R}^d \) and a value \( r > 0 \).
- The spatial transformation \( x^* : \mathbb{R}^d \setminus \{b\} \mapsto \mathbb{R}^d \setminus \{b\} \)

\[
x^* = b + \frac{r^2}{|x - b|^2} (x - b),
\]

is called an inversion through the sphere \( S_{d-1}(b, r) := \{x \in \mathbb{R}^d : |x - b| = r\} \).

Figure: Inversion relative to the sphere \( S_{d-1}(b, r) \).
**Inversion through $S_{d-1}(b, r)$: Key Properties**

Inversion through $S_{d-1}(b, r)$

$$x^* = b + \frac{r^2}{|x-b|^2}(x-b),$$

The following can be deduced by straightforward algebra

- **Self inverse**
  $$x = b + r^2 \frac{(x^* - b)}{|x^* - b|^2}$$

- **Symmetry**
  $$r^2 = |x^* - b||x - b|$$

- **Difference**
  $$|x^* - y^*| = \frac{r^2|x - y|}{|x - b||y - b|}$$

- **Differential**
  $$dx^* = \frac{r^{2d}}{|x - b|^{2d}}dx$$
INVERSION THROUGH $\mathbb{S}_{d-1}(b, r)$: KEY PROPERTIES

- The sphere $\mathbb{S}_{d-1}(c, R)$ maps to itself under inversion through $\mathbb{S}_{d-1}(b, r)$ provided the former is orthogonal to the latter, which is equivalent to $r^2 + R^2 = |c - b|^2$.

- In particular, the area contained in the blue segment is mapped to the area in the red segment and vice versa.
A variant of the sphere inversion transform takes the form

\[ x^\circ = b - \frac{r^2}{|x - b|^2} (x - b), \]

and has properties

- **Self inverse**
  \[ x = b - \frac{r^2}{|x^\circ - b|^2} (x^\circ - b), \]

- **Symmetry**
  \[ r^2 = |x^\circ - b||x - b|, \]

- **Difference**
  \[ |x^\circ - y^\circ| = \frac{r^2|x - y|}{|x - b||y - b|}. \]

- **Differential**
  \[ dx^\circ = \frac{r^{2d}}{|x - b|^{2d}} dx \]
### Sphere Inversion with Reflection

- Fix $b \in \mathbb{R}^d$ and $r > 0$. The sphere $S_{d-1}(c, R)$ maps to itself through $S_{d-1}(b, r)$ providing $|c - b|^2 + r^2 = R^2$.

- However, this time, the exterior of the sphere $S_{d-1}(c, R)$ maps to the interior of the sphere $S_{d-1}(c, R)$ and vice versa. For example, the region in the exterior of $S_{d-1}(c, R)$ contained by blue boundary maps to the portion of the interior of $S_{d-1}(c, R)$ contained by the red boundary.
§11. Spherical hitting distribution
Port’s Sphere hitting distribution

A richer version of the previous theorem:

Theorem (Port (1969))

Define the function

\[ h^\odot(x, y) = \frac{\Gamma\left(\frac{\alpha + d}{2} - 1\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(d\frac{d}{2}\right) \Gamma(\alpha - 1)} \frac{||x|^2 - 1|^{\alpha - 1}}{|x - y|^{\alpha + d - 2}} \]

for \( |x| \neq 1, |y| = 1 \). Then, if \( \alpha \in (1, 2) \),

\[ \mathbb{P}_x(X_{\tau^\odot} \in dy) = h^\odot(x, y)\sigma_1(dy)\mathbf{1}_{|x| \neq 1} + \delta_x(dy)\mathbf{1}_{|x| = 1}, \quad |y| = 1, \]

where \( \sigma_1(dy) \) is the surface measure on \( \mathbb{S}_{d-1} \), normalised to have unit total mass.

Otherwise, if \( \alpha \in (0, 1] \), \( \mathbb{P}_x(\tau^\odot = \infty) = 1 \), for all \( |x| \neq 1 \).
Proof of Port’s sphere hitting distribution

- Write \( \mu_x^\circ(dz) = \mathbb{P}_x(X_{\tau^\circ} \in dz) \) on \( S_{d-1} \) where \( x \in \mathbb{R}^d \setminus S_{d-1} \).
Proof of Port’s sphere hitting distribution

- Write $\mu^\circ_x(dz) = \mathbb{P}_x(X_{\tau^\circ} \in dz)$ on $\mathbb{S}_{d-1}$ where $x \in \mathbb{R}^d \setminus \mathbb{S}_{d-1}$.

- Recall the expression for the resolvent of the stable process in Theorem 1 which states that, due to transience,

$$\int_0^\infty \mathbb{P}_x(X_t \in dy)dt = C(\alpha)|x - y|^{\alpha - d}dy, \quad x, y \in \mathbb{R}^d,$$

where $C(\alpha)$ is an unimportant constant in the following discussion.
Proof of Port’s sphere hitting distribution

- Write $\mu_x^\circ(dz) = \mathbb{P}_x(X_{\tau^\circ} \in dz)$ on $\mathbb{S}_{d-1}$ where $x \in \mathbb{R}^d \setminus \mathbb{S}_{d-1}$.

- Recall the expression for the resolvent of the stable process in Theorem 1 which states that, due to transience,

$$\int_0^\infty \mathbb{P}_x(X_t \in dy) dt = C(\alpha)|x - y|^{\alpha - d} dy, \quad x, y \in \mathbb{R}^d,$$

where $C(\alpha)$ is an unimportant constant in the following discussion.

- The measure $\mu_x^\circ$ is the solution to the ‘functional fixed point equation’

$$|x - y|^{\alpha - d} = \int_{\mathbb{S}_{d-1}} |z - y|^{\alpha - d} \mu(dz), \quad y \in \mathbb{S}_{d-1}.$$

Note that $y \in \mathbb{S}_{d-1}$, so the occupation of $y$ from $x$, will at least see the the process pass through the sphere $\mathbb{S}_{d-1}$ somewhere first (if not $y$).

- With a little work, we can show it is the unique solution in the class of probability measures.
Proof of Port’s sphere hitting distribution

Recall, for $y^* \in S_{d-1}$, from the Riesz representation of the sphere hitting probability,

$$
\frac{\Gamma \left( \frac{d}{2} \right) \Gamma(\alpha - 1)}{\Gamma \left( \frac{\alpha + d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)} = \int_{S_{d-1}} |z^* - y^*|^{\alpha - d} \sigma_1(dz^*).
$$

we are going to manipulate this identity using sphere inversion to solve the fixed point equation first assuming that $|x| > 1$.
Proof of Port’s Sphere Hitting Distribution

Recall, for \( y^* \in S_{d-1} \), from the Riesz representation of the sphere hitting probability,

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\frac{\Gamma \left( \frac{d}{2} \right) \Gamma(\alpha - 1)}{\Gamma \left( \frac{\alpha + d}{2} - 1 \right) \Gamma \left( \frac{\alpha}{2} \right)} = \int_{S_{d-1}} |z^* - y^*|^{\alpha - d} \sigma_1(dz^*).
\]

we are going to manipulate this identity using sphere inversion to solve the fixed point equation first assuming that \(|x| > 1\)

- Apply the sphere inversion with respect to the sphere \( S_{d-1}(x, (|x|^2 - 1)^{1/2}) \) remembering that this transformation maps \( S_{d-1} \) to itself and using

\[
\frac{1}{|z^* - x|^{d-1}} \sigma_1(dz^*) = \frac{1}{|z - x|^{d-1}} \sigma_1(dz)
\]

\(|x|^2 - 1 = |z^* - x||z - x| \quad \text{and} \quad |z^* - y^*| = \frac{|x|^2 - 1|z - y|}{|z - x||y - x|}
\]
PROOF OF PORT’S SPHERE HITTING DISTRIBUTION

Recall, for \( y^* \in S_{d-1} \), from the Riesz representation of the sphere hitting probability,

\[
\frac{\Gamma\left(\frac{d}{2}\right) \Gamma(\alpha - 1)}{\Gamma\left(\frac{\alpha+d}{2} - 1\right) \Gamma\left(\frac{\alpha}{2}\right)} = \int_{S_{d-1}} |z^* - y^*|^{\alpha-d} \sigma_1(dz^*). 
\]

we are going to manipulate this identity using sphere inversion to solve the fixed point equation **first assuming that** \(|x| > 1\)

- **Apply the sphere inversion with respect to the sphere** \( S_{d-1}(x, (|x|^2 - 1)^{1/2}) \) **remembering that this transformation maps** \( S_{d-1} \) **to itself and using**

\[
\frac{1}{|z^* - x|^{d-1}} \sigma_1(dz^*) = \frac{1}{|z - x|^{d-1}} \sigma_1(dz)
\]

\((|x|^2 - 1) = |z^* - x||z - x| \quad \text{and} \quad |z^* - y^*| = \frac{(|x|^2 - 1)|z - y|}{|z - x||y - x|} \]

- **We have**

\[
\frac{\Gamma\left(\frac{d}{2}\right) \Gamma(\alpha - 1)}{\Gamma\left(\frac{\alpha+d}{2} - 1\right) \Gamma\left(\frac{\alpha}{2}\right)} = \int_{S_{d-1}} |z^* - x|^{d-1} |z^* - y^*|^{\alpha-d} \frac{\sigma_1(dz^*)}{|z^* - x|^{d-1}} 
\]

\(= \frac{(|x|^2 - 1)^{\alpha-1}}{|y - x|^{\alpha-d}} \int_{S_{d-1}} \frac{|z - y|^{\alpha-d}}{|z - x|^{\alpha+d-2}} \sigma_1(dz). \)

- **For the case** \(|x| < 1\), **use Riesz–Bogdan–Žak theorem again!** (See exercises).
§12. Spherical entrance/exit distribution
Theorem

Define the function

\[ g(x, y) = \pi^{-(d/2+1)} \Gamma(d/2) \sin(\pi \alpha/2) \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |y|^2|^{\alpha/2}} |x - y|^{-d} \]

for \( x, y \in \mathbb{R}^d \setminus S_{d-1} \). Let

\[ \tau^\oplus := \inf\{ t > 0 : |X_t| < 1 \} \text{ and } \tau_a^\oplus := \inf\{ t > 0 : |X_t| > 1 \}. \]

(i) Suppose that \( |x| < 1 \), then

\[ \mathbb{P}_x(X_{\tau^\oplus} \in dy) = g(x, y)dy, \quad |y| \geq 1. \]

(ii) Suppose that \( |x| > 1 \), then

\[ \mathbb{P}_x(X_{\tau^\oplus} \in dy, \, \tau^\oplus < \infty) = g(x, y)dy, \quad |y| \leq 1. \]
Proof of $B$–$G$–$R$ entrance/exit distribution (I)

Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

$$|x - y|^{\alpha - d} = \int_{|z| \geq 1} |z - y|^{\alpha - d} \mu(dz), \quad |y| > 1 > |x|,$$

with a straightforward argument providing uniqueness.
Appealing again to the potential density and the strong Markov property, it suffices to find a solution to

\[ |x - y|^\alpha - d = \int_{|z| \geq 1} |z - y|^\alpha - d \mu(dz), \quad |y| > 1 > |x|, \]

with a straightforward argument providing uniqueness.

The proof is complete as soon as we can verify that

\[ |x - y|^\alpha - d = c_{\alpha,d} \int_{|z| \geq 1} |z - y|^\alpha - d \left( \frac{1 - |x|^2/|z|^2}{1 - |z|^2/|\alpha/2|^2} |x - z|^{-d} \right) dz \]

for \(|y| > 1 > |x|\), where

\[ c_{\alpha,d} = \pi^{-(1+d/2)} \Gamma(d/2) \sin(\pi\alpha/2). \]
Proof of B–G–R entrance/exit distribution (I)

- Transform $z \mapsto z^\circ$ (sphere inversion with reflection) through the sphere $S_{d-1}(x, (1 - |x|^2)^{1/2})$, noting in particular that

$$|z^\circ - y^\circ| = (1 - |x|^2) \frac{|z - y|}{|z - x||y - x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z - x|^2}{1 - |x|^2} (1 - |z^\circ|^2)$$

and

$$dz^\circ = (1 - |x|^2)^d |z - x|^{-2d} dz, \quad z \in \mathbb{R}^d.$$
Proof of B–G–R Entrance/Exit Distribution (I)

Transform \( z \mapsto z^\circ \) (sphere inversion with reflection) through the sphere \( S_{d-1}(x, (1 - |x|^2)^{1/2}) \), noting in particular that

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|z^\circ - y^\circ| = (1 - |x|^2) \frac{|z - y|}{|z - x||y - x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z - x|^2}{1 - |x|^2} (1 - |z^\circ|^2)
\]

and

\[
dz^\circ = (1 - |x|^2)^d |z - x|^{-2d} dz, \quad z \in \mathbb{R}^d.
\]

For \(|x| < 1 < |y|\),

\[
\int_{|z| \geq 1} |z - y|^{\alpha-d} \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |z|^2|^{\alpha/2}} |x - z|^{-d} dz = |y - x|^{\alpha-d} \int_{|z^\circ| \leq 1} \frac{|z^\circ - y^\circ|^{\alpha-d}}{|1 - |z^\circ|^2|^{\alpha/2}} dz^\circ.
\]
PROOF OF B–G–R ENTRANCE / EXIT DISTRIBUTION (I)

Transform $z \mapsto z^\circ$ (sphere inversion with reflection) through the sphere $S_{d-1}(x, (1 - |x|^2)^{1/2})$, noting in particular that

$$|z^\circ - y^\circ| = (1 - |x|^2) \frac{|z - y|}{|z - x||y - x|} \quad \text{and} \quad |z|^2 - 1 = \frac{|z - x|^2}{1 - |x|^2} (1 - |z^\circ|^2)$$

and

$$dz^\circ = (1 - |x|^2)^d |z - x|^{-2d} dz, \quad z \in \mathbb{R}^d.$$

For $|x| < 1 < |y|,$

$$\int_{|z| \geq 1} |z - y|^{\alpha - d} \frac{1 - |x|^2|\alpha/2}{1 - |z|^2|\alpha/2} |x - z|^{-d} dz = |y - x|^{\alpha - d} \int_{|z^\circ| \leq 1} |z^\circ - y^\circ|^{\alpha - d} \frac{1 - |y|^2|\alpha/2}{1 - |z^\circ|^2|\alpha/2} dz^\circ.$$

Now perform similar transformation $z^\circ \mapsto w$ (inversion with reflection), albeit through the sphere $S_{d-1}(y^\circ, (1 - |y|^2)^{1/2})$.

$$|y - x|^{\alpha - d} \int_{|z^\circ| \leq 1} \frac{|z^\circ - y^\circ|^{\alpha - d}}{1 - |z^\circ|^2|\alpha/2} dz^\circ = |y - x|^{\alpha - d} \int_{|w| \geq 1} \frac{1 - |y|^2|\alpha/2}{1 - |w|^2|\alpha/2} |w - y^\circ|^{-d} dw.$$
PROOF OF B–G–R ENTRANCE/EXIT DISTRIBUTION (I)

Thus far:

\[
\int_{|z| \geq 1} |z-y|^\alpha \cdot \frac{|1 - |x|^2|\alpha/2}{|1 - |z|^2|\alpha/2} |x-z|^{-d} \, dz = |y-x|^\alpha \cdot \int_{|w| \geq 1} \frac{|1 - |y|^2|\alpha/2}{|1 - |w|^2|\alpha/2} |w-y|^\alpha \cdot \frac{|1 - |x|^2|\alpha/2}{|1 - |z|^2|\alpha/2} |x-z|^{-d} \, dw.
\]

Taking the integral in red and decomposition into generalised spherical polar coordinates

\[
\int_{|v| \geq 1} \frac{1}{|1 - |w|^2|\alpha/2} |w-y|^\alpha \cdot \frac{|1 - |x|^2|\alpha/2}{|1 - |z|^2|\alpha/2} |x-z|^{-d} \, dw = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{1}^{\infty} \frac{r^{d-1} \, dr}{|1 - r^2|\alpha/2} \int_{S_{d-1}(0,r)} |z - y|^\alpha \cdot \frac{|1 - |x|^2|\alpha/2}{|1 - |z|^2|\alpha/2} |x-z|^{-d} \, \sigma_r \, (dz)
\]
PROOF OF B–G–R ENTRANCE / EXIT DISTRIBUTION (I)

Thus far:

\[
\int_{|z| \geq 1} |z-y|^\alpha - d \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |z|^2|^{\alpha/2}} |x-z|^{-d} \, dz = |y-x|^\alpha - d \int_{|w| \geq 1} \frac{|1 - |y|^2|^{\alpha/2}}{|1 - |w|^2|^{\alpha/2}} |w-y|^\alpha - d \, dw.
\]

\[\hspace{50mm} \rightarrow \hspace{50mm} \]

Taking the integral in red and decomposition into generalised spherical polar coordinates

\[
\int_{|v| \geq 1} \frac{1}{|1 - |w|^2|^{\alpha/2}} |w-y|^\alpha - d \, dw = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_1^\infty \frac{r^{d-1} \, dr}{|1 - r^2|^{\alpha/2}} \int_{S_{d-1}(0,r)} |z-y|^\alpha - d \, \sigma_r(dz)
\]

\[\hspace{50mm} \rightarrow \hspace{50mm} \]

Poisson’s formula (the probability that a Brownian motion hits a sphere of radius \( r > 0 \)) states that

\[
\int_{S_{d-1}(0,r)} \frac{r^{d-2}(r^2 - |y|^2)}{|z-y|^d} \sigma_r(dz) = 1, \quad |y| < 1 < r.
\]

gives us

\[
\int_{|v| \geq 1} \frac{1}{|1 - |w|^2|^{\alpha/2}} |w-y|^\alpha - d \, dw = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_1^\infty \frac{2r}{(r^2 - 1)^{\alpha/2}(r^2 - |y|^2)} \, dr
\]

\[
= \frac{\pi}{\sin(\alpha \pi/2)} \frac{1}{(1 - |y|^2)^{\alpha/2}}
\]

\[\hspace{50mm} \rightarrow \hspace{50mm} \]

Plugging everything back in gives the result for \(|x| < 1\).
Exercises Set 2
EXERCISES

1. Use the fact that the radial part of a $d$-dimensional ($d \geq 2$) isotropic stable process has MAP $(\xi, \Theta)$, for which the first component is a Lévy process with characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma\left(\frac{1}{2}(-iz + \alpha)\right)}{\Gamma\left(-\frac{1}{2}iz\right)} \frac{\Gamma\left(\frac{1}{2}(iz + d)\right)}{\Gamma\left(\frac{1}{2}(iz + d - \alpha)\right)}, \quad z \in \mathbb{R}.$$ 

to deduce the following facts:

- Irrespective of its point of issue, we have $\lim_{t \to \infty} |X_t| = \infty$ almost surely.
EXERCISES

1. Use the fact that the radial part of a $d$-dimensional ($d \geq 2$) isotropic stable process has MAP $(\xi, \Theta)$, for which the first component is a Lévy process with characteristic exponent given by

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$$

to deduce the following facts:

- Irrespective of its point of issue, we have $\lim_{t \to \infty} |X_t| = \infty$ almost surely.
- By considering the roots of $\Psi$ show that $\exp((\alpha - d)\xi_t), \quad t \geq 0,$
is a martingale.
- Deduce that $|X_t|^{\alpha-d}, \quad t \geq 0,$
is a martingale.

2. Remaining in $d$-dimensions ($d \geq 2$), recalling that

$$
\frac{d\mathbb{P}_x^\circ}{d\mathbb{P}_x}\bigg|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0, x \neq 0,
$$

show that under $\mathbb{P}_x^\circ$, $X$ is absorbed continuously at the origin in an almost surely finite time.
Exercises

3. Recall the following theorem

Theorem
Define the function

\[ g(x, y) = \pi^{-(d/2+1)} \Gamma(d/2) \sin(\pi \alpha/2) \frac{|1 - |x|^2|^{\alpha/2}}{|1 - |y|^2|^{\alpha/2}} |x - y|^{-d} \]

for \( x, y \in \mathbb{R}^d \setminus \mathbb{S}_{d-1} \). Let

\[ \tau^\oplus := \inf\{t > 0 : |X_t| < 1\} \text{ and } \tau^\ominus_a := \inf\{t > 0 : |X_t| > 1\}. \]

(i) Suppose that \( |x| < 1 \), then

\[ \mathbb{P}_x(X_{\tau^\oplus} \in dy) = g(x, y)dy, \quad |y| \geq 1. \]

(ii) Suppose that \( |x| > 1 \), then

\[ \mathbb{P}_x(X_{\tau^\ominus} \in dy, \tau^\ominus < \infty) = g(x, y)dy, \quad |y| \leq 1. \]

Prove (ii) (i.e. \( |x| > 1 \)) from the identity in (i) (i.e. \( |x| < 1 \)).
References


