The portfolio optimization problem by a general bivariate functional of the mean and variance and its solution

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Markowitz's 1952 classical mean variance (MV) measure for optimal portfolio selection has been extensively studied over the years. The MV model relies on the *mean-variance* measure,

$$MV(R) = E(R) - \lambda Var(R), \quad \lambda > 0, \tag{1}$$

where $R = \mathbf{x}^T\mathbf{X}$ is the portfolio return and $\mathbf{X} = (X_1, ..., X_n)^T$ is the vector of random returns with a vector of expectations μ and a covariance matrix Σ, and $\mathbf{x}^T = (x_1, ..., x_n)$ is the vector of weights such that $\mathbf{1}^T\mathbf{x} = 1$, where $\mathbf{1}$ is a vector-column of n ones. $E(R) = \mu^T\mathbf{x}$ and $Var(R) = \mathbf{x}^T\Sigma\mathbf{x}$ are the expected return and the variance of the portfolio return R. The aim of the MV optimal portfolio selection is the maximization of (1). In [1] the maximization of the classical goal function (1) is replaced by the maximization of the functional $MSV(R) = E(R) - \lambda s(Var(R))$, where the function $s(x)$, defined on $[0, \infty)$, is differentiable and positive on $(0, \infty)$. Furthermore, in [2] the maximization solution of a more general functional was obtained, that is based on the ratio of functionals of $E(R)$ and $Var(R)$.

In this paper we consider such a functional in its most general form

$$\rho(R) = F(E(R), Var(R)). \tag{2}$$

The general measure (2) includes important measures such as the mean-variance, value at risk, expected shortfall, tail mean variance, Sharpe ratio, quadratic utility, Neuman-Morgenster utility, and many other celebrated measures. We show an explicit solution to the following optimization problem

$$\mathcal{F}(\mathbf{x})=F(\mu^T\mathbf{x},\mathbf{x}^T\Sigma\mathbf{x}) \to \max, \tag{3}$$

subject to a number of linear constraints

$$B\mathbf{x} = \mathbf{c}. \tag{4}$$

Here B and $\mathbf{c}$ are $m \times n, m < n$, and $m \times 1$ full-rank matrix and vector, respectively, where $\mathbf{c} \neq \mathbf{0}$ with $\mathbf{0}$ the vector-column of m zeros.

The explicit solution takes the form

$$\mathbf{x}^* = \mathbf{x}^0 + \omega^*\mathbf{z}\,, \tag{5}$$

where $\mathbf{x}^0$ and $\mathbf{z}$ are $n \times 1$ vectors that have explicit forms which depend only on $\mu, \mathbf{c}, \Sigma, B$, and not on the functional F, and ω^* is the unique solution of some univariate non-linear equation.

This portfolio optimization problem is closely related to expected utility maximization and two-moments decision models. We observed that all the optimization problems corresponding to the general functional considered here reduce to the same efficient frontier.

Keywords: optimal portfolio selection, expected utility maximization, two-moments decision models, Sharpe ratio

References

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