Cascade Sensitivity Measures

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Abstract

In risk management, an internal model consists of three elements: (i) a random vector of *input risk factors*, $\boldsymbol{X} = (X_1, \dots, X_n)$, (ii) a real valued *aggregation function*, $g \colon \mathbb{R}^n \to \mathbb{R}$ and (iii) the output $Y = g(\boldsymbol{X})$, the random variable obtained by applying the aggregation function to the vector of risk factors. We define a novel sensitivity measure termed *cascade sensitivity* to risk factor X_i by

$$S_i(\boldsymbol{X}, g, \rho) = \frac{\partial}{\partial \varepsilon} \rho\big(g\big(\psi(X_{i,\varepsilon}, \boldsymbol{V})\big)\big)\big|_{\varepsilon=0},$$

where ρ is a law-invariant risk measure summarising the output Y and $X_{i,\varepsilon}$ a distortion of the risk factor X_i. For given i, we represent the vector of input risk factors by $\boldsymbol{X} = \psi(X_i, \boldsymbol{V})$, for a function $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ and a $(n-1)$-dimensional random vector $\boldsymbol{V}$ independent of X_i, where the dependence of ψ and $\boldsymbol{V}$ on i is suppressed [1]. The proposed cascade sensitivity measure captures the direct impact of the distorted (stressed) individual risk factor on the output, as well as indirect effects via the function ψ of other risk factors that are dependent with the one being stressed. In this way, the dependence between risk factors is explicitly taken into account.

For the most common risk measures in industry, Value-at-Risk, $\mathrm{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R} \,|\, P(Z \leq z) \geq \alpha\}$, $0 < \alpha < 1$, and Expected Shortfall, $\mathrm{ES}_\alpha(Z) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(Z)\mathrm{d}u$, $0 < \alpha < 1$, the cascade sensitivity is, under continuity assumptions [2, 3], given by

$$S_i(\boldsymbol{X}, g, \mathrm{VaR}_\alpha) = E\Big((g \circ \psi)_1(X_i, \boldsymbol{V})\, \frac{\partial}{\partial \varepsilon} X_{i,\varepsilon}\big|_{\varepsilon=0} \,\Big|\, Y = \mathrm{VaR}_\alpha(Y)\Big), \tag{1}$$

$$S_i(\boldsymbol{X}, g, \mathrm{ES}_\alpha) = E\Big((g \circ \psi)_1(X_i, \boldsymbol{V})\, \frac{\partial}{\partial \varepsilon} X_{i,\varepsilon}\big|_{\varepsilon=0} \,\Big|\, Y \geq \mathrm{VaR}_\alpha(Y)\Big), \tag{2}$$

where $(g \circ \psi)_1$ denotes the derivative of $g \circ \psi$ with respect to its first argument.

The above representation of the cascade sensitivity requires the knowledge of the aggregation function's gradient which might not be readily available in practice. Thus, we provide representations relying on additional evaluations of the model under alternative distributional assumptions. This is of particular interest in applications where the distribution of the output is typically determined via simulation methods.

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Proposition 1. *For an input risk factor X_i with continuous and unimodal distribution with mode m_i, we define the distortion $X_{i,\varepsilon} = X_i + \varepsilon(X_i - m_i)$. Under suitable continuity assumptions,*

1. *the cascade sensitivity for the VaR to X_i is given by*

$$S_i(\boldsymbol{X}, g, VaR_\alpha) = \frac{\alpha - \tilde{H}(H^{-1}(\alpha))}{h(H^{-1}(\alpha))}$$
$$= \frac{1}{h(H^{-1}(\alpha))}\Big[\alpha - E\Big(\frac{f_{\tilde{X}_i}(X_i)}{f_{X_i}(X_i)}\boldsymbol{I}_{\{Y \le H^{-1}(\alpha)\}}\Big)\Big],$$

2. *the cascade sensitivity for the ES to X_i is given by*

$$S_i(\boldsymbol{X}, g, ES_\alpha) = \frac{1}{1-\alpha}\Big[E\big((\tilde{Y} - H^{-1}(\alpha))_+\big) - E\Big(\big(Y - H^{-1}(\alpha)\big)_+\Big)\Big]$$
$$= \frac{1}{1-\alpha}\Big[E\Big(\frac{f_{\tilde{X}_i}(X_i)}{f_{X_i}(X_i)}\big(Y - H^{-1}(\alpha)\big)_+\Big) - E\Big(\big(Y - H^{-1}(\alpha)\big)_+\Big)\Big].$$

$\tilde{H}$ denotes the distribution of $\tilde{Y} = g(\psi(\tilde{X}_i, \boldsymbol{V}))$, where $\tilde{X}_i$ is a random variable comonotonic to X_i with distribution $\tilde{F}_i(x) = F_i(x) - (x - m_i)f_i(x)$, $x \in \mathbb{R}$.

Compared to (1) and (2), the representations in Proposition 1 of the cascade sensitivity do not require the knowledge of the aggregation function's gradient. The first representation, for both VaR and ES, can be calculated with one additional evaluation of the model, replacing input risk factor X_i with $\tilde{X}_i$. The second representation in Proposition 1 requires one single simulated sample starting from the baseline model, combined with a simple form of importance sampling, without the need to simulate under different model specifications or to explicitly study the properties of the aggregation function. This makes the proposed method attractive for practical applications.

We provide generalisations of Proposition 1 to different stresses of input risk factors with multi-modal distributions as well as to distortion risk measures and illustrate its applicability through numerical examples.

Keywords: Sensitivity analysis, risk management, Value-at-Risk, Expected Shortfall.

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