An Optimal Proportional-Stop-Loss Reinsurance Contracts From Insurer's and Reinsurer's Viewpoints

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Extended Abstract

Suppose random claim X_i has cumulative distribution function $F(x)$, and survival function $\bar{F}(x)$. Moreover, suppose that random claim X_i can be decomposed to the sum of the insurer portion (Y_i) and reinsurer portion $(I(X_i))$, i.e., $X_i = Y_i + I(X_i)$. Now consider the combination of the proportional and excess of loss reinsurance strategies, such as proportional-excess-loss reinsurance

$$Y_i = \alpha \min (X_i, M). \tag{1}$$

This form of reinsurance strategy is called proportional excess of loss reinsurance and is a version of the reinsurance from Centeno (1985). But this article focuses on estimating two unknown parameters the new strategy (1) by taking into account both parties (i.e., insurer's and reinsurer's companies). More precisely, unknown parameters α and M in the proportional excess of loss reinsurance strategy shown in Equation (1) can be estimated in two steps. First, estimate the parameters such that the expected utility of the insurer (or reinsurer) is maximized. Next, use the estimated parameters from the insurer and reinsurer as target estimators. Then develop a Bayesian estimator with respect to the doubly-balanced loss function for each parameter so that the expected surplus of the insurer and reinsurer are maximized.

The following extends the ordinary balanced loss function (with one given target estimator introduced by Zellner (1994) and improved by Jafari et al., 2006) to a doubly-balanced loss function with two target estimators δ_0 and δ_1. $L_{\rho,\omega_1,\omega_2,\delta_0,\delta_1}(\xi,\delta) = \omega_1\rho(\delta_0,\delta)+\omega_2\rho(\delta_1,\delta)+(1-\omega_1-\omega_2)\rho(\xi,\delta)$, where $\omega_1 \in [0,1)$ and $\omega_2 \in [0,1)$ are weights which satisfy $\omega_1+\omega_2 < 1$ and $\rho(\cdot,\cdot)$ is an arbitrary and given loss function. The Bayesian estimator for prior π and under the doubly-balanced loss function with square error loss (i.e., $\rho(\xi,\delta) = (\xi-\delta)^2$) is $\delta_{\pi,\omega_1,\omega_2}(x) = \omega_1\delta_0(x) + \omega_2\delta_1(x) + (1-\omega_1-\omega_2)E_\pi(\xi\,|x)$. The following theorem shows that the proportional-excess-loss reinsurance minimizes the variance of the retained risk in some situations.

Theorem 1. *Suppose $I(X)$ and $I_N(X)$ are the reinsurer contribution under an arbitrary reinsurance strategy and the proportional-excess-loss reinsurance for random claim X, respectively. Moreover, suppose that $E(I(X)) = E(I_N(X))$; $P(I(X) \geq I_N(X)|X \leq M) = 1$; $P(I(X) \geq I_N(X)|X \geq M\&X - I(X) \leq M) = 1$;and $P(I(X) \leq I_N(X)|X \geq M\&X - I(X) \geq M) = 1$. Then $Var(X - I(X)) \geq Var(X - I_N(X))$.*

Theorem (1) provides conditions under which variance of the insurer contribution under proportional-excess-loss reinsurance is less than under other reinsurance strategies. Excess of loss and proportional reinsurance strategies do not satisfy Theorem (1) conditions. Therefore, the above finding does not contradict with Bowers et al. (1997).

The following theorem compares proportional-excess-loss reinsurance with the proportional reinsurance and the excess of loss reinsurance strategies for stochastic dominance.

Theorem 2. *Suppose $I_N(X)$ is the contribution of reinsurance against random claim X under the proportional-excess-loss reinsurance. Moreover, suppose that $I_P(X)$ ($I_E(X)$) is the contribution of reinsurance against random claim X under the proportional (or the excess of loss) reinsurance strategies. Then $P(X - I_N(X) \leq X - I_P(X)) = P(X - I_N(X) \leq X - I_E(X))) = 1$.*

From Theorem (2), one may conclude that, for all $p \in (0,1)$, $\mathrm{VaR}[X - I_N(X);p] \leq \mathrm{VaR}[X - I_E(X);p]$ $(\mathrm{VaR}[X - I_N(X);p] \leq \mathrm{VaR}[X - I_P(X);p])$ and $\mathrm{TVaR}[X - I_N(X);p] \leq \mathrm{TVaR}[X - I_E(X);p]$.

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Suppose U_t and U_t^*, respectively restate the surplus of the insurer and reinsurer under a reinsurance strategy $U_t = u_0 + (1+\theta_0)E(\sum_{i=1}^{N(t)} Y_i) - \sum_{i=1}^{N(t)} Y_i$and $U_t^* = u_0^* + \pi_1(t) - \sum_{i=1}^{N(t)} I(X_i)$,where u_0 and u_0^* are the initial wealth of the insurer and reinsurer, random variable Y_i is the insurer portion of random claim X_i, θ_0 is the safety factor, $\pi_1(t)$ is premium of the reinsurance strategy in time t and $N(t)$ is the Poisson process with intensity λ.

Under exponential utility (i.e., $u(x) = -e^{-\beta_0 x}$), the following theorem (3) provides two estimators for α and M.

Theorem 3. *Suppose U_t and U_t^*, respectively stand for the surplus of insurer and reinsurer for the proportional-excess-loss reinsurance strategy* (1)*. Then, under exponential utility (i.e., $u(x) = -e^{-\beta_0 x}$), the following $\hat{\alpha}_0$ and $\hat{M}_0$ maximize the expected exponential utility the insurer's terminal surplus U_t and $\hat{\alpha}_1$ and $\hat{M}_1$ maximize the expected exponential utility of the reinsurer's terminal surplus is U_t^*.*

$$
\begin{aligned}
0 &= \hat{\alpha}_0\beta_0\hat{M}_0 - \ln(1+\theta_0), \quad \& \quad 0 = \int_{\hat{M}_1}^{\infty} \beta_1\hat{\alpha}_1 e^{-\beta_1(x-\hat{\alpha}_1\hat{M}_1)}dF(x) - \beta_1(1+\theta)\hat{\alpha}_1(1-F(\hat{M}_1)), \\
0 &= \beta_0(1+\theta_0)\lambda t\int_0^{\hat{M}_0} xdF(x) + \beta_0(1+\theta_0)\lambda t\hat{M}_0\bar{F}(\hat{M}_0) - \lambda\beta_0 t\int_0^{\hat{M}_0} xe^{\hat{\alpha}_0\beta_0 x}dF(x) + \lambda\beta_0 t\hat{M}_0 e^{\hat{\alpha}_0\beta_0\hat{M}_0}\bar{F}(\hat{M}_0), \\
0 &= \int_0^{\hat{M}_1} \beta_1 x e^{\beta_1(1-\hat{\alpha}_1)x}dF(x) + \int_{\hat{M}_1}^{\infty} \beta_1\hat{M}_1 e^{\beta_1(x-\hat{\alpha}_1\hat{M}_1)}dF(x) + \beta_1(1+\theta_1)\int_0^{\infty}(x1_{[0,\hat{M}_1)}(x) + \hat{M}_1 1_{[\hat{M}_1,\infty)}(x))dF(x).
\end{aligned}
$$

Up to now, two parameters are estimated from insurer's and reinsurer's viewpoint separately. Now, to integrate the above results and define an optimal reinsurance strategy of both parties, the following Bayesian estimator under the doubly-balanced loss function has been employed.

Theorem 4. *Suppose $Z_1, \cdots, Z_n|(\theta,\alpha,M)$ are a sequence of i.i.d. random variables with common density function $f_{Z|(\theta,\alpha,M)}(z)$. Moreover, suppose that $\pi_1(\Theta)$, $\pi_2(\mathcal{A})$, and $\pi_3(\mathcal{M})$ are prior distributions for θ, α, and M, respectively. Then, the Bayesian estimators for α and M for the square error doubly-balanced loss function, prior distribution π, and target estimators $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{M}_0$, $\hat{M}_1$ are*

$$\hat{\alpha}_{\pi,\omega_1,\omega_2} = \omega_1\hat{\alpha}_0 + \omega_2\hat{\alpha}_1 + (1-\omega_1-\omega_2)E_\pi(\mathcal{A}|z), \quad \& \quad \hat{M}_{\pi,\omega_1,\omega_2} = \omega_1\hat{M}_0 + \omega_2\hat{M}_1 + (1-\omega_1-\omega_2)E_\pi(\mathcal{M}|z).$$

The last section of this section provides two numerical examples to show how the above findings can be applied in practice. It develops **(i)** estimators for α and M, $\hat{\alpha}_0$ and $\hat{M}_0$, so that insurer wealth is maximized; **(ii)** estimators for α and M, $\hat{\alpha}_1$ and $\hat{M}_1$, so that reinsurer's wealth is maximized; **(iii)** Bayesian estimators for α and M for the square error doubly-balanced loss function for prior distributions $\alpha \sim Beta(2,2)$ and $M \sim Exp(2)$, and target estimators $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{M}_0$, $\hat{M}_1$.

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