Self-similar Markov processes
Problems

§Exercise Set 1

EXERCISES

1. Suppose that X is a stable process in any dimension (including the case of a Brownian motion). Show that $|X|$ is a positive self-similar Markov process.
2. Suppose that B is a one-dimensional Brownian motion. Prove that

$$\frac{B_t}{x}\mathbf{1}_{(\underline{B}_t>0)}, \qquad t \geq 0,$$

is a martingale, where $\underline{B}_t = \inf_{s\leq t} B_s$.
3. Suppose that X is a stable process with two-sided jumps
   - Show that the range of the ascending ladder process H, say $\texttt{range}(H)$ has the property that it is equal in law to $c \times \texttt{range}(H)$.
   - Hence show that, up to a multiplicative constant, the Laplace exponent of H satisfies $k(\lambda) = \lambda^{\alpha_1}$ for $\alpha_1 \in (0,1)$ (and hence the ascending ladder height process is a stable subordinator).
   - Use the fact that, up to a multiplicative constant

$$\Psi(z) = |\theta|^{\alpha}(\mathrm{e}^{\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}) = \hat{\kappa}(\mathrm{i}z)\kappa(-\mathrm{i}z)$$

   to deduce that

$$\kappa(\theta) = \theta^{\alpha\rho} \text{ and } \hat{\kappa}(\theta) = \theta^{\alpha\hat{\rho}}.$$

   and that $0 < \alpha\rho, \alpha\hat{\rho} < 1$
   - What kind of process corresponds to the case that $\alpha\rho = 1$?

EXERCISES

4. Suppose that $(X, \mathrm{P}_x), x > 0$ is a positive self-similar Markov process and let $\zeta = \inf\{t > 0 : X_t = 0\}$ be the lifetime of X. Show that $\mathrm{P}_x(\zeta < \infty)$ does not depend on x and is either 0 for all $x > 0$ or 1 for all $x > 0$.
5. Suppose that X is a symmetric stable process in dimension one (in particular $\rho = 1/2$) and that the underlying Lévy process for $|X_t|\mathbf{1}_{(t<\tau^{\{0\}})}$, where $\tau^{\{0\}} = \inf\{t > 0 : X_t = 0\}$, is written ξ. (Note the indicator is only needed when $\alpha \in (1, 2)$ as otherwise X does not hit the origin.) Show that (up to a multiplicative constant) its characteristic exponent is given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + 1))}{\Gamma(\frac{1}{2}(\mathrm{i}z + 1 - \alpha))}, \qquad z \in \mathbb{R}.$$

[Hint!] Think about what happens after X first crosses the origin and apply the Markov property as well as symmetry. You will need to use the law of the overshoot of X below the origin given a few slides back.

Exercises

6. Use the previous exercise to deduce that the MAP exponent underlying a stable process with two sided jumps is given by

$$\begin{bmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.

Exercises Set 2

Exercises

1. Use the fact that the radial part of a d-dimensional ($d \geq 2$) isotropic stable process has MAP (ξ, Θ), for which the first component is a Lévy process with characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + d))}{\Gamma(\frac{1}{2}(\mathrm{i}z + d - \alpha))}, \qquad z \in \mathbb{R}.$$

to deduce the following facts:

- Irrespective of its point of issue, we have $\lim_{t\to\infty} |X_t| = \infty$ almost surely.

EXERCISES

1. Use the fact that the radial part of a d-dimensional ($d \geq 2$) isotropic stable process has MAP (ξ, Θ), for which the first component is a Lévy process with characteristic exponent given by

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   to deduce the following facts:
   - Irrespective of its point of issue, we have $\lim_{t\to\infty} |X_t| = \infty$ almost surely.
   - By considering the roots of Ψ show that

     $$\exp((\alpha - d)\xi_t), \qquad t \geq 0,$$

     is a martingale.
   - Deduce that

     $$|X_t|^{\alpha - d}, \qquad t \geq 0,$$

     is a martingale.

2. Remaining in d-dimensions ($d \geq 2$), recalling that

$$\left.\frac{\mathrm{d}\mathbb{P}^{\circ}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0,$$

   show that under $\mathbb{P}^{\circ}$, X is absorbed continuously at the origin in an almost surely finite time.

Exercises

3. Recall the following theorem

Theorem

Define the function

$$g(x,y) = \pi^{-(d/2+1)}\,\Gamma(d/2)\,\sin(\pi\alpha/2)\,\frac{\left|1-|x|^2\right|^{\alpha/2}}{\left|1-|y|^2\right|^{\alpha/2}}\,|x-y|^{-d}$$

for $x, y \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$. *Let*

$$\tau^{\oplus} := \inf\{t > 0 : |X_t| < 1\} \text{ and } \tau_a^{\ominus} := \inf\{t > 0 : |X_t| > 1\}.$$

(i) Suppose that $|x| < 1$, *then*

$$\mathbb{P}_x(X_{\tau^\ominus} \in \mathrm{d}y) = g(x,y)\mathrm{d}y, \qquad |y| \geq 1.$$

(ii) Suppose that $|x| > 1$, *then*

$$\mathbb{P}_x(X_{\tau^\oplus} \in \mathrm{d}y,\ \tau^{\oplus} < \infty) = g(x,y)\mathrm{d}y, \qquad |y| \leq 1.$$

Prove (ii) (i.e. $|x| > 1$) from the identity in (i) (i.e. $|x| < 1$).