Self-similar Markov processes
Part I: One dimension

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A more thorough set of lecture notes can be found here:
https://arxiv.org/abs/1707.04343
Other related material found here
https://arxiv.org/abs/1511.06356
https://arxiv.org/abs/1706.09924

CONTENTS

§1. Quick review of Lévy processes

(Killed) Lévy process

- $(\xi_t, t \geq 0)$ is a (killed) Lévy process if it has stationary and independents with RCLL paths (and is sent to a cemetery state after and independent and exponentially distributed time).

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- Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula

$$\mathbf{E}[\mathrm{e}^{\mathrm{i}\theta\cdot\xi_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R}^d,$$

where,

$$\Psi(\theta) = q + \mathrm{i}\mathrm{a}\cdot\theta + \frac{1}{2}\theta\cdot\mathbf{A}\theta + \int_{\mathbb{R}^d}(1 - \mathrm{e}^{\mathrm{i}\theta\cdot x} + \mathrm{i}(\theta\cdot x)\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x),$$

where $\mathrm{a} \in \mathbb{R}$, $\mathbf{A}$ is a $d \times d$ Gaussian covariance matrix and Π is a measure satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$. Think of Π as the intensity of jumps in the sense of

$$\mathbf{P}(X \text{ has jump at time } t \text{ of size } \mathrm{d}x) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

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- In one dimension the path of a Lévy process can be monotone, in which case it is called a *subordinator* and we work with the Laplace exponent

$$\mathbf{E}[\mathrm{e}^{-\lambda\xi_t}] = \mathrm{e}^{-\Phi(\lambda)t}, \qquad t \geq 0$$

where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

Lévy process: One dimension

Two examples in one dimension:

- **Stable subordinator** $(\xi_t, t \geq 0)$ is a subordinator which satisfies the additional scaling property: For $c > 0$

$$\text{under } \mathbb{P}, \text{ the law of } (c\xi_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P},$$

where $\alpha \in (0,1)$. We have

$$\Phi(\lambda) = \lambda^{\alpha}, \qquad \lambda \geq 0, \qquad \text{and} \qquad \Pi(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}} \mathrm{d}x, \qquad x > 0.$$

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- **Hypgergeometric Lévy process:** For $\beta \leq 1, \gamma \in (0, 1), \hat{\beta} \geq 0, \hat{\gamma} \in (0, 1)$

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

The Lévy measure has a density with respect to Lebesgue measure which is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\[2ex] -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$.

Lévy process: One dimension

- If ξ has a characteristic exponent Ψ then necessarily

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

where κ and $\hat{\kappa}$ are Bernstein functions, e.g.

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- The factorisation has a physical interpretation:
  - range of the κ-subordinator agrees with the range of $\sup_{s\leq t} \xi_s$, $t \geq 0$
  - range $\hat{\kappa}$-subordinator agrees with the range of $-\inf_{s\leq t} \xi_s$, $t \geq 0$.

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- Note if $\delta > 0$, then $\mathbf{P}(\xi_{\tau_x^+} = x) > 0$, where $\tau_x^+ = \inf\{t > 0 : \xi_t > x\}$, $x > 0$.

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- Note if $\delta > 0$, then $\mathbf{P}(\xi_{\tau_x^+} = x) > 0$, where $\tau_x^+ = \inf\{t > 0 : \xi_t > x\}$, $x > 0$.
- We have already seen the hypergeometric example

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \qquad \times \qquad \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

Hitting points

- We say that ξ *can hit a point* $x \in \mathbb{R}$ if

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Theorem (Kesten (1969)/Bretagnolle (1971))

Suppose that ξ is not a compound Poisson process. Then ξ can hit points if and only if

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z < \infty.$$

If the Kesten-Bretagnolle integral test is satisfied, then

$$\mathbb{P}(\tau^{\{x\}} < \infty) = \frac{u(x)}{u(0)},$$

where $\tau^{\{x\}} = \inf\{t > 0 : \xi_t = x\}$, providing we can compute the inversion

$$u(x) = \int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)} \mathrm{d}z$$

for some $c \in \mathbb{R}$.

§2. Self-similar Markov processes

Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a rssMp if there exists an index $\alpha \in (0, \infty)$ such that for all $c > 0$ and $x \in \mathbb{R}^d$,

$$(cZ_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (Z_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

Some of your best friends are ssMp

- Write $\mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma})$ for the Normal distribution with mean $\mathbf{0} \in \mathbb{R}^d$ and correlation (matrix) $\mathbf{\Sigma}$. The moment generating function of $X_t \sim \mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma}t)$ satisfies, for $\theta \in \mathbb{R}^d$,
$$\mathbf{E}[\mathrm{e}^{\theta \cdot X_t}] = \mathrm{e}^{t\theta^{\mathrm{T}}\mathbf{\Sigma}\theta/2} = \mathrm{e}^{(c^{-2}t)(c\theta)^{\mathrm{T}}\mathbf{\Sigma}(c\theta)/2} = E[\mathrm{e}^{\theta \cdot cX_{c^{-2}t}}].$$

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- Thinking about the stationary and independent increments of Brownian motion, this can be used to show that $\mathbb{R}^d$**-Brownian motion:** is a ssMp with $\alpha = 2$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}$-Brownian motion:

- Write $\underline{X}_t := \inf_{s \leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process.

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- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c \inf_{s \leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u \leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.

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- Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ issued from $x > 0$ with index 2.

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- $Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.

Some of your best friends are ssMp

Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Consider $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process.
- Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$.

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- Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$.
- Note that $|X_t|$, $t \geq 0$ is a Bessel-d process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on $[0, \infty)$. Once can check this by e.g. considering scaling properties of their transition semi-groups.

Some of your best friends are ssMp

Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Note when $d = 3$, $|X_t|, t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s \to \infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x \left[\frac{B_t}{x} \mathbf{1}_{(\underline{B}_t > 0)} \mathbf{1}_{(A)} \right]$$

where $A \in \sigma\{B_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}_x^{\uparrow}).$$

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- We want to find more exotic examples as most of the previous examples have been extensively studied through existing theories (of Brownian motion and continuous semi-martingales).
- All of the previous examples are functional transforms of Brownian motion and have made use of the scaling and Markov properties and (in some cases) isotropic distributional invariance.
- If we replace Brownain motion by an α-stable process, a Lévy process that has scale invariance, then all of the functional transforms still produce new examples of self-similar Markov processes.

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- The characteristic exponent $\Psi(\theta) := -t^{-1}\log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

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  where $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$
- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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- which by stationary and independent increments is equivalent to saying $(cX_{c^{-\alpha}t}, t \geq 0) \stackrel{d}{=} (X_t, t \geq 0)$ when $X_0 = 0$,
- or equivalently is equivalent to saying $(cX^{(x)}_{c^{-\alpha}t}, t \geq 0) \stackrel{d}{=} (X^{(cx)}_t, t \geq 0)$, where we have indicated the point of issue as an additional index.

STABLE PROCESS PATH PROPERTIES

index	jumps	path	recurrence/transience
$\alpha \in (0,1)$			transient
$\rho = 0$	$-$	monotone decreasing	$\lim_{t\to\infty} X_t = -\infty$
$\rho = 1$	$+$	monotone increasing	$\lim_{t\to\infty} X_t = \infty$
$\rho \in (0,1)$	$+,-$	bounded variation	$\lim_{t\to\infty} \lvert X_t\rvert = \infty$
$\alpha = 1$			recurrent
$\rho = \frac{1}{2}$	$+,-$	unbounded variation	$\limsup_{t\to\infty} \lvert X_t\rvert = \infty$, $\liminf_{t\to\infty} \lvert X_t\rvert = 0$
$\alpha \in (1,2)$			recurrent
$\alpha\rho = 1$	$-$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $-\liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t = \infty$
$\alpha\rho = \alpha - 1$	$+$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $-\liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t = \infty$
$\alpha\rho \in (\alpha-1, 1)$	$+,-$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $-\liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t = \infty$

Your new friends

Suppose $X = (X_t : t \geq 0)$ is within the assumed class of α-stable processes in one-dimension and let $\underline{X}_t = \inf_{s \leq t} X_s$.

Your new friends are:

- $Z = X$
- $Z = X - (-x \wedge \underline{X})$, $x > 0$.
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- $Z = X\mathbf{1}_{(\underline{X}>0)}$
- $Z = |X|$ providing $\rho = 1/2$
- What about $Z =$"X conditioned to stay positive"?

CONDITIONED α-STABLE PROCESSES

- Recall that each Lévy processes, $\xi = \{\xi_t : t \geq 0\}$, enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant, $\Psi_\xi(\theta) := t^{-1} \log \mathbf{E}[\mathrm{e}^{\mathrm{i}\theta\xi_t}]$ respects the factorisation

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where ν is a measure satisfying $\int_{(0,\infty)} (1 \wedge x)\nu(\mathrm{d}x) < \infty$.

Conditioned α-stable processes

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$$\kappa(\lambda) = q + \mathrm{a}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x), \qquad \lambda \geq 0$$

  where ν is a measure satisfying $\int_{(0,\infty)} (1 \wedge x)\nu(\mathrm{d}x) < \infty$.

- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of ξ and of $-\xi$ respectively.

CONDITIONED α-STABLE PROCESSES

- Recall that each Lévy processes, $\xi = \{\xi_t : t \geq 0\}$, enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant, $\Psi_\xi(\theta) := t^{-1} \log \mathbf{E}[e^{i\theta\xi_t}]$ respects the factorisation
$$\Psi_\xi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \qquad \theta \in \mathbb{R},$$
where κ and $\hat{\kappa}$ are Bernstein functions. That is e.g. κ takes the form
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where ν is a measure satisfying $\int_{(0,\infty)} (1 \wedge x)\nu(dx) < \infty$.
- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of ξ and of $-\xi$ respectively.
- In the case of α-stable processes, up to a multiplicative constant,
$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \qquad \lambda \geq 0.$$

Conditioned α-stable processes

- Associated to the descending ladder subordinator $\hat{\kappa}$ is its potential measure $\hat{U}$, which satisfies

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda x} \hat{U}(\mathrm{d}x) = \frac{1}{\hat{\kappa}(\lambda)}, \qquad \lambda \geq 0.$$

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- It can be shown that for a Lévy process which satisfies $\limsup_{t\to\infty} \xi_t = \infty$, for $A \in \sigma(\xi_u : u \leq t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A|\underline{X}_{t+s} > 0) = \mathbb{E}_x\left[\frac{\hat{U}(X_t)}{\hat{U}(x)}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right]$$

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- In the α-stable case $\hat{U}(x) \propto x^{\alpha\hat{\rho}}$
  [Note in the excluded case that $\alpha = 2$ and $\rho = 1/2$, i.e. Brownian motion, $\hat{U}(x) = x$.]

Conditioned α-stable processes

- For $c, x > 0$, $t \geq 0$ and appropriately bounded, measurable and non-negative f, we can write,

$$
\begin{aligned}
&\mathbb{E}^{\uparrow}_x[f(\{cX_{c^{-\alpha}s} : s \leq t\})] \\
&\quad = \mathbb{E}\left[f(\{cX^{(x)}_{c^{-\alpha}s} : s \leq t\}) \frac{(X^{(x)}_{c^{-\alpha}t})^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}} \mathbf{1}_{(\underline{X}^{(x)}_{c^{-\alpha}t} \geq 0)} \right] \\
&\quad = \mathbb{E}\left[f(\{X^{(cx)}_s : s \leq t\} \frac{(X^{(cx)}_t)^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}} \mathbf{1}_{(\underline{X}^{(cx)}_t \geq 0)} \right] \\
&\quad = \mathbb{E}^{\uparrow}_{cx}[f(\{X_s : s \leq t\})].
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&\qquad = \mathbb{E}\left[f(\{X^{(cx)}_s : s \leq t\} \frac{(X^{(cx)}_t)^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}} \mathbf{1}_{(\underline{X}^{(cx)}_t \geq 0)}\right] \\
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\end{aligned}
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- This also makes the process $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, a self-similar Markov process on $[0, \infty)$.
- Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process (the analogue to the Brownian case).

§3. Lamperti Transform

NOTATION

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log \mathbf{E}(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

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$$-\log \mathbf{E}(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

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- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

Lamperti transform for POSITIVE ssMp

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) *$\zeta^{(x)} = \infty$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,*

(2) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$, or*

(3) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

LAMPERTI TRANSFORM FOR POSITIVE SSMP

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}\mathbf{1}_{(t < x^\alpha I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^\alpha I_\infty$, with index α.

LAMPERTI TRANSFORM FOR POSITIVE SSMP

$$
\begin{array}{ccc}
(Z, \mathbb{P}_x)_{x>0} \text{ pssMp} & & (\xi, \mathbf{P}_y)_{y\in\mathbb{R}} \text{ killed Lévy} \\
Z_t = \exp(\xi_{S(t)}), & \leftrightarrow & \xi_s = \log(Z_{T(s)}), \\
S \text{ a random time-change} & & T \text{ a random time-change}
\end{array}
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$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \quad \leftrightarrow \quad \left\{\begin{array}{c} \xi\to\infty \text{ or } \xi \text{ oscillates} \\ \xi\to-\infty \\ \xi \text{ is killed} \end{array}\right.$$

§4. Positive self-similar Markov processes

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.

Stable process killed on entry to $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z^*_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.

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- Write $\xi^* = \{\xi^*_t : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.

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- Write $\xi^* = \{\xi_t^* : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.
- Let's try and decode the characteristics of ξ^*.

Stable process killed on entry to $(-\infty, 0)$

- We know that the α-stable process experiences downward jumps at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x<0.$$

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- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z^*_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right) = \frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})(Z^*_{t-})^{-\alpha}.$$

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- On the other hand, the Lamperti transform says that on $\{t<\zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}} = q^*(Z^*_t)^{-\alpha}.$$

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- Comparing gives us

$$q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

Stable process killed on entry to $(-\infty, 0)$

- Referring again to the Lamperti transform, we know that, under $\mathbb{P}_1$ (so that $\xi^*_0 = 0$ almost surely),

$$Z^*_{\zeta-} = X_{\tau_0^- -} = e^{\xi^*_{\mathbf{e}_{q^*}}},$$

where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.

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where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.
- This motivates the computation
$$\mathbb{E}_1[(Z^*_{\zeta-})^{i\theta}] = \mathbf{E}_0[e^{i\theta\xi^*_{\mathbf{e}_{q^*}-}}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \qquad \theta \in \mathbb{R},$$
where Ψ^* is the characteristic exponent of ξ^*.

Stable process killed on entry to $(-\infty, 0)$

Remembering the "overshoot-undershoot" distributional law at first passage (well known in the literature for Lévy processes c.f. the quintuple law - Chapter 7 of my book) and deduce that, for all $v \in [0,1]$,

$$
\begin{aligned}
&\mathbb{P}_1(X_{\tau_0^- -} \in \mathrm{d}v) \\
&\quad = \hat{\mathbb{P}}_0(1 - X_{\tau_1^+ -} \in \mathrm{d}v) \\
&\quad = \frac{\sin(\alpha\hat{\rho}\pi)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\int_0^\infty \int_0^\infty \mathbf{1}_{(y \le 1 \wedge v)} \frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} \mathrm{d}u\mathrm{d}y \right) \mathrm{d}v \\
&\quad = \frac{\sin(\alpha\hat{\rho}\pi)}{\pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\int_0^1 \mathbf{1}_{(y \le v)} v^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1} \mathrm{d}y \right) \mathrm{d}v,
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$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

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\end{aligned}$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

Note: more generally (which you will need for an exercise later):

$$\begin{aligned}
&\mathbb{P}_1(-X_{\tau_0^-} \in \mathrm{d}u,\, X_{\tau_0^- -} \in \mathrm{d}v)\\
&\quad = \frac{\sin(\alpha\hat\rho\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}\left(\int_0^\infty \mathbf{1}_{(y\le 1\wedge v)}\frac{(1-y)^{\alpha\hat\rho-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}}\mathrm{d}y\right)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

Stable process killed on entry to $(-\infty, 0)$

We are led to the conclusion that

$$
\begin{aligned}
&\frac{q*}{\Psi^*(\theta)} \\
&= \frac{\sin(\alpha\hat\rho\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}\int_0^1 (1-y)^{\alpha\hat\rho-1}\int_0^\infty \mathbf{1}_{(y\leq v)} v^{\mathrm{i}\theta-\alpha\hat\rho-1}\left(1-\frac{y}{v}\right)^{\alpha\rho-1} \mathrm{d}v\mathrm{d}y \\
&= \frac{\sin(\alpha\hat\rho\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}\int_0^1 (1-y)^{\alpha\hat\rho-1} y^{\mathrm{i}\theta-\alpha\hat\rho}\mathrm{d}y\frac{\Gamma(\alpha\hat\rho-\mathrm{i}\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-\mathrm{i}\theta)} \\
&= \frac{\Gamma(\alpha\hat\rho-\mathrm{i}\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat\rho+\mathrm{i}\theta)\Gamma(\alpha\hat\rho)\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)\Gamma(\alpha\hat\rho)\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)},
\end{aligned}
$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution $w = y/v$ has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants q^* and K, we come to rest at the following result:

Stable process killed on entry to $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying killed Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

STABLE PROCESSES CONDITIONED TO STAY POSITIVE

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t > 0)}\mathbf{1}_{(A)}\right] = \mathbf{E}_0\left[\mathrm{e}^{\alpha\hat{\rho}\xi^*_\tau}\mathbf{1}_{(\tau < \mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

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- Noting that $\Psi^*(-\mathrm{i}\alpha\hat{\rho}) = 0$, the change of measure constitutes an Esscher transform at the level of ξ^*.

Stable processes conditioned to stay positive

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Theorem

The underlying Lévy process, $\xi^{\uparrow}$, that appears through the Lamperti transform applied to $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, has characteristic exponent given by

$$\Psi^{\uparrow}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1 + \alpha\hat{\rho} + \mathrm{i}z)}{\Gamma(1 + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^{\uparrow}(z) = \Psi^*(z - \mathrm{i}\alpha\hat{\rho})$, $z \in \mathbb{R}$ so that $\Psi^{\uparrow}(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = \mathbf{E}_0[\xi^{\uparrow}_1] > 0$ so that $\lim_{t\to\infty}\xi^{\uparrow}_t = \infty$.

Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.

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- However, there is another root of the equation

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namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.

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  namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.
- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

  is a unit-mean Martingale, which can also be used to construct an Esscher transform:

$$\Psi^{\downarrow}(z) = \Psi^*(z - \mathrm{i}(1 - \alpha\hat{\rho})) = \Psi^{\downarrow}(z) = \frac{\Gamma(1 + \alpha\rho - \mathrm{i}z)}{\Gamma(1 - \mathrm{i}z)} \frac{\Gamma(\mathrm{i}z + \alpha\hat{\rho})}{\Gamma(\mathrm{i}z)}.$$

Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.
- However, there is another root of the equation

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)} = 0,$$

  namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.
- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

  is a unit-mean Martingale, which can also be used to construct an Esscher transform:

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- The choice of notation is pre-emptive since we can also check that $\Psi^{\downarrow}(0) = 0$ and $\Psi^{\downarrow\prime}(0) < 0$ so that if $\xi^{\downarrow}$ is a Lévy process with characteristic exponent $\Psi^{\downarrow}$, then $\lim_{t\to\infty} \xi^{\downarrow}_t = -\infty$.

Reverse engineering

- What now happens if we define for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}^{\downarrow}_x(A) = \mathbf{E}_0\left[\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right] = \mathbb{E}_x\left[\frac{X_t^{(1-\alpha\hat{\rho})}}{x^{(1-\alpha\hat{\rho})}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

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- In the same way we checked that $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}_x^{\downarrow})$, $x > 0$ is a pssMp.

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  where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.
- In the same way we checked that $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is a pssMp.
- In an appropriate sense, it turns out that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is the law of a stable process conditioned to continuously approach the origin from above.

ξ^*, $\xi^\uparrow$ AND $\xi^\downarrow$

- The three examples of pssMp offer quite striking underlying Lévy processes
- Is this exceptional?

Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t > 0)} \, \mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbf{1}_{(t < T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1,2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

Censored stable processes

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overleftrightarrow{\xi}$. Then $\overleftrightarrow{\xi}$ is equal in law to $\xi^{**} \oplus \xi^{C}$, with

- *ξ^{**} equal in law to ξ^{*} with the killing removed,*
- *ξ^{C} a compound Poisson process with jump rate $q^{*} = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$.*

Moreover, the characteristic exponent of $\overleftrightarrow{\xi}$ is given by

$$\overleftrightarrow{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(1-\alpha+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

THE RADIAL PART OF A STABLE PROCESS

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written ξ, then it characteristic exponent is given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + 1))}{\Gamma(\frac{1}{2}(\mathrm{i}z + 1 - \alpha))}, \qquad z \in \mathbb{R}.$$

HYPERGEOMETRIC LÉVY PROCESSES (REMINDER)

Definition (and Theorem)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\left\{ \ \beta \leq 2, \ \gamma, \hat{\gamma} \in (0,1) \ \hat{\beta} \geq -1, \text{ and } 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0 \ \right\}$$

there exists a (killed) Lévy process, henceforth refered to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\[2ex] -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$, for $|z|<1$, ${}_2F_1(a,b;c;z) := \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

§5. Entrance Laws

Starting from zero

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STARTING FROM ZERO

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- This should ring alarm bells when we look at the Lamperti transform

$$Z_t^{(x)}\mathbf{1}_{(t<\zeta^{(x)})} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\} = \exp\{\xi_{\varphi(x^{-\alpha}t)} + \log x\}, \qquad t \geq 0,$$

- On the one hand $\log x \downarrow -\infty$, which is the point of issue of ξ, but

$$\varphi(x^{-\alpha}t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha(\xi_u + \log x)}\mathrm{d}u > t\},$$

  meaning that we are sampling the Lévy process over a longer and longer time horizon.

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- In that case, for any sequence of times $0 < t_1 \leq t_2 \leq \cdots \leq t_n < \infty$ and $y_1, \cdots, y_n \in (0, \infty)$, $n \in \mathbb{N}$, the Markov property gives us

$$\begin{aligned}
&\mathbb{P}_0(Z_{t_1} \in \mathrm{d}y_1, \cdots, Z_{t_n} \in \mathrm{d}y_n) \\
&\quad := \lim_{x\downarrow 0} \mathbb{P}_x(Z_{t_1} \in \mathrm{d}y_1, \cdots, Z_{t_n} \in \mathrm{d}y_n) \\
&\quad = \lim_{x\downarrow 0} \mathbb{P}_x(Z_{t_1} \in \mathrm{d}y_1)\mathbb{P}_{y_1}(Z_{t_2-t_1} \in \mathrm{d}y_2, \cdots, Z_{t_n-t_2} \in \mathrm{d}y_n) \\
&\quad = \mathbb{P}_0(Z_{t_1} \in \mathrm{d}y_1)\mathbb{P}_{y_1}(Z_{t_2-t_1} \in \mathrm{d}y_2, \cdots, Z_{t_n-t_2} \in \mathrm{d}y_n).
\end{aligned}$$

  When the limit exists, it implies the existence of $\mathbb{P}_0$ as limit of $\mathbb{P}_x$ as $x \downarrow 0$, in the sense of convergence of finite-dimensional distributions.

Starting from zero

- We would like a stronger sense of convergence e.g. we would like

$$\mathbb{E}_0[f(Z_s : s \leq t)] := \lim_{x \to 0} \mathbb{E}_x[f(Z_s : s \leq t)]$$

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  for an appropriate measurable function on cadlag paths of length t.
- The right setting to discuss *distributional convergence* is with respect to so-called *Skorokhod topology*.
- ROUGHLY: There is a metric on cadlag path space which does a better job of measuring how "close" two paths are than e.g. the uniform functional metric.
- This metric induces a topology (the Skorokhod topology). From this topology, we build a measurable space around the space of cadlag paths.
- Think of $\mathbb{P}_x$, $x > 0$ as a family of measures on this space and we want weak convergence "$\mathbb{P}_0 := \lim_{x \to 0} \mathbb{P}_x$" on this space.

Starting from zero

Theorem

Suppose that $(\xi, \mathbf{P}_x)$, $x \in \mathbb{R}$ is the Lévy process (not a compound Poisson process) underlying the pssMp $(Z, \mathbb{P}_x)$, $x > 0$. The limit $\mathbb{P}_0 := \lim_{x \to 0} \mathbb{P}_x$ exists in the sense of convergence with respect to the Skorokhod topology if and only if $\mathbf{E}_0(H_1^+) < \infty$ (H^+ is the ascending ladder process of ξ). Under the assumption that $\mathbb{E}(\xi_1) > 0$, for any positive measurable function f and $t > 0$,

$$\mathbb{E}_0(f(Z_t)) = \frac{1}{-\alpha \hat{\mathbf{E}}_0(\xi_1)} \hat{\mathbf{E}}_0 \left(\frac{1}{I_\infty} f\left(\left(\frac{t}{I_\infty} \right)^{1/\alpha} \right) \right),$$

where $I_\infty = \int_0^\infty \mathrm{e}^{\alpha \xi_t} \mathrm{d}t$ and $(\xi, \hat{\mathbf{P}}_0)$ is equal in law to $(-\xi, \mathbf{P}_0)$.

Recurrent extension

- The previous construction has insisted that Z is a *pssMp* with $\zeta = \infty$ a.s. But what about the case that $\zeta < \infty$ a.s.

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- A cadlag strong Markov process, $\vec{Z} := \{\vec{Z}_t : t \geq 0\}$ with probabilities $\{\vec{P}_x, x \geq 0\}$, is a *recurrent extension* of Z if, for each $x > 0$, the origin is not an absorbing state $\vec{P}_x$-almost surely and $\{\vec{Z}_{t\wedge\vec{\zeta}} : t \geq 0\}$ under $\vec{P}_x$ has the same law as (Z, P_x), where

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$$\vec{\zeta} = \inf\{t > 0 : \vec{Z}_t = 0\}.$$

Theorem

If $\zeta < \infty$ a.s. and $X_{\zeta -} = 0$, then there exists a unique recurrent extension of Z which leaves 0 continuously if and only if there exists a $\beta \in (0, \alpha)$ such

$$\mathbf{E}_0(e^{\beta \xi_1}) = 1.$$

Here, as usual, α is the index of self-similarity.

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§6. Real valued self-similar Markov processes

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- What can we say about $\mathbb{R}$-valued self-similar Markov processes.
- This requires us to first investigate Markov Additive (Lévy) Processes

Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}, i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,

  $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with law $\mathbf{P}_{0,i}$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J, the process ξ has the representation

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j\in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$ with $i \neq j$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Otherwise define $U_{i,i} \equiv 0$, for each $i \in E$.
- Write $G(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{ij}(z)$.
- Let

$$\boldsymbol{\Psi}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \mathbf{Q} \circ G(z),$$

  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbf{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{\Psi}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

  (when it exists).

Dual MAP

- Thanks to irreducibility, the Markov chain J necessarily has a stationary distribution. We denote it by the vector $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_N)$.
- Each MAP has a dual process, also a MAP, with probabilities $\hat{\mathbf{P}}_{x,i}$, $x \in \mathbb{R}$, $i \in E$, determined by the dual characteristic matrix exponent (when it exists),

$$\hat{\boldsymbol{\Psi}}(z) := \operatorname{diag}\big(-\Psi_1(-z), \cdots, -\Psi_N(-z)\big) + \hat{\boldsymbol{Q}} \circ \boldsymbol{G}(-z)^{\mathrm{T}},$$

  where $\hat{\boldsymbol{Q}}$ is the time-reversed Markov chain J,

$$\hat{q}_{i,j} = \frac{\pi_j}{\pi_i} q_{j,i}, \qquad i, j \in E.$$

  Note that the latter can also be written $\hat{\boldsymbol{Q}} = \boldsymbol{\Delta}_\pi^{-1} \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{\Delta}_{\boldsymbol{\pi}}$, where $\boldsymbol{\Delta}_{\boldsymbol{\pi}} = \operatorname{diag}(\boldsymbol{\pi})$.
- When it exists,

$$\hat{\boldsymbol{\Psi}}(z) = \boldsymbol{\Delta}_\pi^{-1} \boldsymbol{\Psi}(-z)^{\mathrm{T}} \boldsymbol{\Delta}_\pi,$$

  showing that

$$\pi_i \hat{\mathbf{E}}_{0,i}\left[\mathrm{e}^{\mathrm{i}z\xi_t}, J_t = j\right] = \pi_j \mathbf{E}_{0,j}\left[\mathrm{e}^{-\mathrm{i}z\xi_t}, J_t = i\right].$$

Lemma

The time-reversed process $\left(\left(\xi_{(t-s)-} - \xi_t, J_{(t-s)-}\right), s \le t\right)$ *under* $\mathbf{P}_{0,\boldsymbol{\pi}}$ *is equal in law to* $((\xi_s, J_s), s \le t)$ *under* $\hat{\mathbf{P}}_{0,\boldsymbol{\pi}}$.

Lamperti-Kiu transform

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where
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and
$$T_0 = |x|^{-\alpha} \int_0^\infty e^{\alpha\xi(u)} \mathrm{d}u.$$
- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.

LAMPERTI-KIU TRANSFORM

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- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within a special class of rssMps) is also true.

Entrance at zero

- Given the Lamperti-Kiu representation

$$Z_t = \mathrm{e}^{\xi(\tau(|x|^{-\alpha}t))+\log|x|}J(\tau(|x|^{-\alpha}t)) \qquad 0 \leq t < T_0,$$

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  it is clear that we can think of a similar construction from zero to the case of pssMp.
- We need to construct a stationary version of the pair (ξ, J) which is indexed by $\mathbb{R}$ and pinned at space-time point $(-\infty, \infty)$.
- Just like the theory of Lévy processes, by observing the range of the process (ξ_t, J_t) $t \ge 0$, **only** at the points of its new suprema, we see a process (H_t^+, J_t^+), $t \ge 0$, which is also a MAP, where H^+ is has increasing paths.

Entrance at zero

Theorem

Assume that Z is a conservative real self-similar Markov process. Moreover, suppose that the MAP $((\xi, \Theta), \mathbf{P})$, associated with Z through the Lamperti-Kiu transform, is such that ξ is not concentrated on a lattice and its ascending ladder height process H which satisfies $\mathbf{E}_{0,\boldsymbol{\pi}}(H_1) < \infty$. Then $\mathbb{P}_0 := \lim_{x\downarrow 0} \mathbb{P}_x$ exists, in the sense of convergence of on the Skorokhod space, under which Z leaves the origin continuously. Conversely, if $\mathbf{E}_{0,\boldsymbol{\pi}}(H_1) = \infty$, then this limit does not exist. Under the additional assumption that $\mathbf{E}_{0,\boldsymbol{\pi}}(\xi_1) > 0$, for any positive measurable function f and $t > 0$,

$$\mathbb{E}_0(f(Z_t)) = \frac{1}{-\alpha \hat{\mathbf{E}}_{0,\boldsymbol{\pi}}(\xi_1)} \sum_{i=\pm 1} \pi_i \hat{\mathbf{E}}_{0,i}\left(\frac{1}{I_\infty} f\left(i\left(\frac{t}{I_\infty}\right)^{1/\alpha}\right)\right), \tag{3}$$

where $I_\infty = \int_0^\infty \exp\{\alpha\xi_s\}\mathrm{d}s$, and $\hat{\mathbf{E}}_{x,i}$, $x \in \mathbb{R}$, $i = \pm 1$.

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- When $\alpha \in (0,1]$, the process never hits the origin a.s.
- When $\alpha \in (1,2)$, the process is absorbs at the origin a.s.
- The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\[2ex] \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1,\alpha)$. Note a matrix $\boldsymbol{A}$ in this context is arranged with the ordering

$$\begin{pmatrix} A_{1,1} & A_{1,-1} \\ A_{-1,1} & A_{-1,-1} \end{pmatrix}.$$

Esscher transform for MAPs

- If $\mathbf{\Psi}(z)$ is well defined then it has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues.
- Furthermore, the corresponding right-eigenvector $\mathbf{v}(z) = (v_1(z), \cdots, v_N(z))$ has strictly positive entries and may be normalised such that $\pi \cdot \mathbf{v}(z) = 1$.

Theorem

Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, and

$$M_t := \mathrm{e}^{\gamma\xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \qquad t \geq 0,$$

for some $\gamma \in \mathbb{R}$ such that $\chi(\gamma)$ is defined. Then, M_t, $t \geq 0$, is a unit-mean martingale. Moreover, under the change of measure

$$\mathrm{d}\mathbf{P}^{\gamma}_{0,i}\Big|_{\mathcal{G}_t} = M_t \, \mathrm{d}\mathbf{P}_{0,i}\Big|_{\mathcal{G}_t}, \qquad t \geq 0,$$

the process (ξ, J) remains in the class of MAPs with new exponent given by

$$\mathbf{\Psi}_\gamma(z) = \mathbf{\Delta}_v(\gamma)^{-1}\mathbf{\Psi}(z+\gamma)\mathbf{\Delta}_v(\gamma) - \chi(\gamma)\mathbf{I}.$$

Here, $\mathbf{I}$ is the identity matrix and $\mathbf{\Delta}_v(\gamma) = \mathrm{diag}(\boldsymbol{v}(\gamma))$.

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- With all of the above

$$\lim_{t\to\infty} \frac{\xi_t}{t} = \chi'(0) \qquad \text{a.s.}$$

Esscher and the stable-MAP

- For the MAP that underlies the stable process $D = (-1, \alpha)$, it can be checked that $\det\Psi(\alpha - 1) = 0$ i.e. $\chi(\alpha - 1) = 0$, which makes

$$\boldsymbol{\Psi}^{\circ}(z) = \boldsymbol{\Delta}^{-1}\boldsymbol{\Psi}(z + \alpha - 1)\boldsymbol{\Delta}$$

$$= \begin{bmatrix} -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\[2ex] \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},$$

where $\boldsymbol{\Delta} = \mathrm{diag}(\sin(\pi\alpha\hat{\rho}), \sin(\pi\alpha\rho))$.

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- When $\alpha \in (1,2)$, $\chi'(0) < 0$ (because the stable process touches the origin a.s.) and $\boldsymbol{\Psi}^\circ(z)$-MAP drifts to $+\infty$.

Riesz-Bogdan-Zak transform

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is an α-stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^{\alpha-1} \mathbf{1}_{(t<\tau^{\{0\}})}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $\mathbf{\Psi}^\circ(z)$.

What is the Ψ°-MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- When $\alpha \in (0,1)$, $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ has the law of the the stable process conditioned to absorb continuously at the origin in the sense,

$$\mathbb{P}^\circ_y(A) = \lim_{a \to 0} \mathbb{P}_y(A, t < T_0 \,|\, \tau_{(-a,a)} < \infty),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$,
  $\tau_{(-a,a)} = \inf\{t > 0 : |X_t| < a\}$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.

- When $\alpha \in (1,2)$, $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ has the law of the stable process conditioned to avoid the origin in the sense

$$\mathbb{P}^\circ_y(A) = \lim_{s \to \infty} \mathbb{P}_y(A \,|\, T_0 > t + s),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.

§Exercise Set 1

Exercises

1. Suppose that X is a stable process in any dimension (including the case of a Brownian motion). Show that $|X|$ is a positive self-similar Markov process.
2. Suppose that B is a one-dimensional Brownian motion. Prove that

$$\frac{B_t}{x}\mathbf{1}_{(\underline{B}_t>0)}, \qquad t \geq 0,$$

is a martingale, where $\underline{B}_t = \inf_{s\leq t} B_s$.

3. Suppose that X is a stable process with two-sided jumps
   - Show that the range of the ascending ladder process H, say $\texttt{range}(H)$ has the property that it is equal in law to $c \times \texttt{range}(H)$.
   - Hence show that, up to a multiplicative constant, the Laplace exponent of H satisfies $k(\lambda) = \lambda^{\alpha_1}$ for $\alpha_1 \in (0,1)$ (and hence the ascending ladder height process is a stable subordinator).
   - Use the fact that, up to a multiplicative constant

   $$\Psi(z) = |\theta|^{\alpha}(\mathrm{e}^{\pi i\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi i\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}) = \hat{\kappa}(\mathrm{i}z)\kappa(-\mathrm{i}z)$$

   to deduce that

   $$\kappa(\theta) = \theta^{\alpha\rho} \text{ and } \hat{\kappa}(\theta) = \theta^{\alpha\hat{\rho}}.$$

   and that $0 < \alpha\rho, \alpha\hat{\rho} < 1$
   - What kind of process corresponds to the case that $\alpha\rho = 1$?

Exercises

4. Suppose that (X, P_x), $x > 0$ is a positive self-similar Markov process and let $\zeta = \inf\{t > 0 : X_t = 0\}$ be the lifetime of X. Show that $\mathrm{P}_x(\zeta < \infty)$ does not depend on x and is either 0 for all $x > 0$ or 1 for all $x > 0$.
5. Suppose that X is a symmetric stable process in dimension one (in particular $\rho = 1/2$) and that the underlying Lévy process for $|X_t|\mathbf{1}_{(t<\tau^{\{0\}})}$, where $\tau^{\{0\}} = \inf\{t > 0 : X_t = 0\}$, is written ξ. (Note the indicator is only needed when $\alpha \in (1, 2)$ as otherwise X does not hit the origin.) Show that (up to a multiplicative constant) its characteristic exponent is given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + 1))}{\Gamma(\frac{1}{2}(\mathrm{i}z + 1 - \alpha))}, \qquad z \in \mathbb{R}.$$

[Hint!] Think about what happens after X first crosses the origin and apply the Markov property as well as symmetry. You will need to use the law of the overshoot of X below the origin given a few slides back.

EXERCISES

6. Use the previous exercise to deduce that the MAP exponent underlying a stable process with two sided jumps is given by

$$\begin{bmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\[2ex] \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.