Wiener-Hopf factorization, excursion theory, and related topics for processes with stationary and independent increments

Contents

1 Introduction

The subject is part of what is known as fluctuation theory and aims at giving information of extrema of processes with stationary and independent increments over a, possibly infinite,

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period of time. Originally, it was relied heavily on complex-analytic techniques. The aim of these lectures is to present a fully probabilistic approach. Processes with stationary and independent increments in discrete time are known as random walks. In continuous time they are known as Lévy processes. To develop the gist of the probabilistic approach it is necessary that we start from discrete time first.

We assume that the reader is familiar with the notions of stopping times, the strong Markov property, and Laplace and Fourier transforms of Borel measures. We shall use the terminology "random element" for a random variable (a measurable function on some measurable space) that takes values in another measurable space. The terminology indicates that the second measurable space can (and will) be much more general than a Euclidean space; for example, it can be a space of functions. If X, Y are random elements with values in the same space (but defined, possibly, on different domains) we write $X \stackrel{\text{(d)}}{=} Y$ to mean that they have the same law (=distribution).

Here is a ridiculously simple but important observation. Let $Z_1, Z_2, \ldots$ be i.i.d. random elements of the same space. Let τ be the hitting time of a certain set A, that is,

$$\tau = \inf\{n \geq 1 : Z_n \in A\}.$$

Notice that τ is a stopping time (because $\{\tau = n\} = \{Z_1, \ldots, Z_{n-1} \notin A, Z_n \in A\}$) that takes values in $\mathbb{N} \cup \{\infty\}$. If $\mathbb{P}(A) > 0$ then $\tau < \infty$ a.s. and τ is a geometric random variable: $\mathbb{P}(\tau = n) = (1 - \mathbb{P}(A))^{n-1}\mathbb{P}(A)$. But the thing that I wish to point out is that (provided $\mathbb{P}(A) > 0$)

$$(Z_1, \ldots, Z_{\tau-1}) \text{ is independent of } Z_\tau$$

and that, of course, $Z_{\tau-1} \in A^c$ but $Z_\tau \in A$ a.s. This simple fact is known as découpage de Lévy. It is this combination of *independence* and *disjointness* (one variable lives in A, the other in A^c) that makes the Wiener-Hopf factorization work. This is to be seen. Moreover, we can realize (simulate!) the (law of the) random element $(Z_1, \ldots, Z_\tau)$ as follows. Let μ be the law of Z_1. Let $\nu, Z'', Z'_1, Z'_2, \ldots$, be independent such that ν is geometric in $\mathbb{N}$ with parameter $\mathbb{P}(A)$; Z'' has law $\mu(\cdot|A^c)$; each Z'_i has law $\mu(\cdot|A)$. Then

$$(Z'_1, \ldots, Z'_{\nu-1}, Z'') \stackrel{\text{(d)}}{=} (Z_1, \ldots, Z_{\tau-1}, Z_\tau).$$

See Exercise 3.

Note one more thing: whereas τ is a stopping time, $\tau - 1$ is not: if we observe the first m random elements are outside A we cannot decide, on the basis of this observation alone, whether the $m + 1$ element is in A. If σ is a general stopping time then we know that

$$(Z_1, \ldots, Z_\sigma) \text{ is independent of } Z_{\sigma+1}. \tag{1}$$

See Exercise 4. In view of this, we can appreciate the découpage de Lévy more: it says something about one step before a stopping time. (The catch is, of course, that τ is not just a stopping time but also a hitting time.)

2 Fine structure of random sums (andante)

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables in $\mathbb{R}$. Let

$$X_n = \xi_1 + \cdots + \xi_n, \quad n \geq 1,$$

with $X_0 = 0$. We are interested in

$$\overline{X}_n = \max_{1 \le t \le n} X_t, \quad \underline{X}_n = \min_{1 \le t \le n} X_t.$$

Maxima and minima are harder than sums. We shall replace a maximum by a sum of random elements over random indices. Let

$$\alpha = \inf\{n \ge 1 : X_n > 0\}.$$

This is a stopping time with values in $\mathbb{N} \cup \{\infty\}$. Introduce shift θ such that

$$\xi_n \circ \theta = \xi_{n+1}.$$

Then $\xi_n \circ \theta^k = \xi_{n+k}$ and $X_n \circ \theta^k = X_{n+k} - X_k$. If $\alpha < \infty$ we can define θ^α and then we can see that

$$\alpha \circ \theta^\alpha = \inf\{n > \alpha : X_n > X_\alpha\}, \qquad \text{on } \alpha < \infty.$$

We can thus define the *iterates*

$$\alpha = \alpha_1 < \alpha_2 < \alpha_3 \cdots$$

of the α recursively by

$$\alpha_1 = \alpha, \quad \alpha_{k+1} = \inf\{n > \alpha_k : X_n > X_{\alpha_k}\} = \inf\{n > \alpha_k : X_n > \overline{X}_{n-1}\} \tag{2}$$

See Exercise 6 for the second equality. Note that the sequence $\alpha_1, \alpha_2, \ldots$ is either infinite or finite. Indeed, if $\alpha = \infty$ has positive probability, the sequence will be finite. We shall thus be careful about the possibility. Now define the α-counting process, that is,

$$L(t) := \sum_{k=1}^{\infty} \mathbf{1}_{\alpha_k \le t} = \sup\{k \ge 1 : \alpha_k \le t\}. \tag{3}$$

See Exercise 7. Clearly, $L(\cdot)$ is unbounded iff the sequence (α_k) is infinite. Observe $L(\alpha_k) = k$ (if $\alpha_k < \infty$) and that, for all t,

$$\alpha_{L(t)} \le t < \alpha_{L(t)+1}.$$

Hence

$$\text{if } \alpha_i \le t < \alpha_{i+1} \text{ then } i = L(t). \tag{4}$$

If the sequence X_n represents performance indices in a certain endurance game, then the times $\alpha_1, \alpha_2, \ldots$ can be thought of as record times. We start with $X_0 = 0$ performance. At times α_k we have the first occurrence of a performance index of value strictly larger than all previous ones. Hence it easy to see that

$$\overline{X}_t = X_{\alpha_{L(t)}}. \tag{5}$$

See Exercise 8. But now we can (and we will) take advantage of the strong Markov property which, in particular, implies that $X_{\alpha_k} - X_{\alpha_{k-1}}$ are independent. Let us express that in terms of cycles. Define the k-th cycle as the random element

$$\mathcal{C}(k) = (\xi_n,\, \alpha_{k-1} < n \le \alpha_k).$$

We have also defined $\mathcal{C}(1)$ if we use the convention $\alpha_0 = 0$. Note that[1]

$$\mathcal{C}(1),\, \mathcal{C}(2),\, \mathcal{C}(3), \ldots \text{ are i.i.d. killed at rate } \mathbb{P}(\alpha = \infty). \tag{6}$$

Indeed, if $\alpha = \infty$ then the sequence consists of only one cycle (of infinite length). On the other hand, conditional on $\alpha_1 < \infty$, we have[2] $(\xi_1, \ldots, \xi_{\alpha_1}) \perp\!\!\!\perp (\xi_{\alpha_1+1}, \ldots, \xi_{\alpha_2})$ by the strong Markov property. And so on. Thus, if $\mathbb{P}(\alpha < \infty) = 1$ the sequence $\mathcal{C}(1), \mathcal{C}(2), \mathcal{C}(3), \ldots$ is an infinite sequence of i.i.d. random elements. If $\mathbb{P}(\alpha < \infty) < 1$ we have a finite sequence $\mathcal{C}(1), \mathcal{C}(2), \ldots, \mathcal{C}(K)$ where K is the first k such that $\alpha_k = \infty$. Then the first $K-1$ cycles have finite lengths; the last cycle has infinite length.

Let next T be an independent geometric$_0$ random variable with parameter p:

$$\mathbb{P}(T = n) = (1-p)^n p, \quad n = 0, 1, 2, \ldots$$

Our first observation is probabilistic in nature:

Lemma 1.

$$(\alpha_{L(T)},\, \overline{X}_T) \perp\!\!\!\perp (T - \alpha_{L(T)},\, X_T - \overline{X}_T).$$

Proof. If we think of tossing a fair coin with probability of heads equal to p, then T represents the number of tails before the first head. We can realize T by means of a sequence $\delta_1, \delta_2, \ldots$ of i.i.d. Bernoulli random variables with $\mathbb{P}(\delta_n = 1) = p$ if we set $T = \inf\{n \in \mathbb{N} : \delta_n = 1\} - 1 = \sup\{n \in \mathbb{N} : \delta_n = 0\}$. Instead of the i.i.d. ξ_n, $n \in \mathbb{N}$, consider the i.i.d. (ξ_n, δ_n), $n \in \mathbb{N}$ and redefine the cycles

$$\mathcal{C}(k) = (\xi_n, \delta_n,\, \alpha_{k-1} < n \le \alpha_k).$$

Since the α_k are still stopping times for the new sequence (ξ_n, δ_n), $n \in \mathbb{N}$, we have that (6) still holds. Let now I be the index of the first cycle containing a head. In other words, if we define $f(\mathcal{C}(k)) := \sum_{\alpha_{k-1} < n \le \alpha_k} \delta_n$, we have $I = \inf\{k \ge 1 : f(\mathcal{C}(k)) \ne 0\}$. Hence

$$\mathcal{C}(I-1) \perp\!\!\!\perp \mathcal{C}(I).$$

We can easily see that (Exercise 10)

$$I = L(T) + 1. \tag{7}$$

Hence

$$\mathcal{C}(L(T)) \perp\!\!\!\perp \mathcal{C}(L(T)+1).$$

But $(\alpha_{L(T)}, X_{\alpha_{L(T)}})$ is a function of $\mathcal{C}(L(T))$ and $(T - \alpha_{L(T)}, X_T - X_{\alpha_{L(T)}})$ is a function of $\mathcal{C}(L(T)+1)$. Hence

$$(\alpha_{L(T)}, X_{\alpha_{L(T)}}) \perp\!\!\!\perp (T - \alpha_{L(T)}, X_T - X_{\alpha_{L(T)}}).$$

But $X_{\alpha_{L(T)}} = \overline{X}_T$. $\square$

[1] If $Z_1, Z_2, \ldots$ is a sequence of random elements, the expression "the sequence $Z_1, Z_2, \ldots$ killed at rate p" is taken to be equivalent to the sentence "consider the sequence $Z_1, Z_2, \ldots, Z_{\gamma-1}$", where γ is an independent geometric random variable with parameter p, i.e. $\mathbb{P}(\gamma = k) = (1-p)^{k-1}p$. If $p = 0$ then $\gamma = \infty$ and so the killed sequence is the original, infinite, sequence: killing at rate 0 means no killing.

[2] The notation $X \perp\!\!\!\perp Y$ means that X and Y are independent random elements.

Our next observation is deterministic in nature. It says that, for each t, the quantity $(t - \alpha_{L(t)}, X_t - \overline{X}_t)$ can be expressed as a deterministic function of the random walk reversed at t. The random walk reversed at t is the sequence

$$\widehat{X}_n := X_t - X_{t-n}, \quad 0 \le n \le t.$$

(Note that $\widehat{X}_n$, $0 \le n \le t$, actually depends on t but we shall omit this from the notation.) The relation is as follows:

Lemma 2.

$$\begin{aligned} & X_t - \overline{X}_t = \underline{\widehat{X}}_t \\ & \alpha_{L(t)} = \sup\{n \le t : \, X_n > \overline{X}_{n-1}\} \\ & t - \alpha_{L(t)} = \sup\{n \le t : \, \widehat{X}_n \le \underline{\widehat{X}}_{n-1}\} \end{aligned}$$

$$\boxed{\textbf{A picture here is worth a thousand words}} \tag{8}$$

Proof. The first equality is obvious. The second equality follows from (2) and the second equality in (3). The last equality is left as an important exercise. (See Exercise 12 below.) $\square$

A comparison between the two last equalities of Lemma 2 and the fact that $\alpha_{L(t)}$ was obtained via iterates of the stopping time α makes it clear that $t - \alpha_{L(t)}$ is also obtainable via iterates of another stopping time, defined as:

$$\beta = \inf\{n \ge 1 : \, X_n \le 0\}.$$

Just like α, we define the iterates of β

$$\beta = \beta_1 < \beta_2 < \cdots$$

and the β-counting process

$$M(t) := \sum_{k=1}^{\infty} \mathbf{1}_{\beta_k \le t} = \sup\{k \ge 1 : \, \beta_k \le t\}.$$

It then follows that

$$\beta_{M(t)} \le t < \beta_{M(t)+1},$$
$$\beta_{M(t)} = \sup\{n \le t : \, X_n \le X_{n-1}\},$$
$$\underline{X}_t = X_{\beta_{M(t)}}.$$

Except that the expression we derived in the last part of Lemma 2 involves the reversed walk rather than the original. Using then the obvious notation (put $\widehat{\beta}$ instead of β when using $\widehat{X}$ instead of X, etc.), we have

Corollary 1.

$$t - \alpha_{L(t)} = \widehat{\beta}_{\widehat{M}(t)}.$$

See also Exercise 5 on the notion of dual stopping times.

Our third observation is again probabilistic.

Lemma 3.

$$(t-\alpha_{L(t)},\, X_t-\overline{X}_t) \overset{\text{(d)}}{=} (\beta_{M(t)},\, \underline{X}_t)$$

Proof. By Lemma 2 and Corollary 1 we have the algebraic relation

$$(t-\alpha_{L(t)},\, X_t-\overline{X}_t) = (\widehat{\beta}_{\widehat{M}(t)},\, \underline{\widehat{X}}_t).$$

Since he reversed walk has the same law as the walk, i.e.,

$$(X_0, X_1, \ldots, X_t) \overset{\text{(d)}}{=} (\widehat{X}_0, \widehat{X}_1, \ldots, \widehat{X}_t),$$

we have $(\widehat{\beta}_{\widehat{M}(t)},\, \underline{\widehat{X}}_t) \overset{\text{(d)}}{=} (\beta_{M(t)},\, \underline{X}_t)$. □

Theorem 1 (path decomposition)**.** *Let X_n, $n \geq 0$, be any random walk in $\mathbb{R}$ with i.i.d. increments and T an independent geometric$_0$ time: $\mathbb{P}(T=n) = (1-p)^n p$. Then*

$$(T,\, X_T) \overset{\text{(d)}}{=} (\alpha_{L(T)},\, \overline{X}_T) \dot{+} (\beta_{M(T)},\, \underline{X}_T),$$

where $\underline{I}$ am using the notation $X_1 \dot{+} X_2$ to denote a random variable whose law is the same as the law of a random variable X_1 and an independent random variable X_2.

Proof. By Lemma 1,

$$(T,\, X_T) \overset{\text{(d)}}{=} (\alpha_{L(T)},\, \overline{X}_T) \dot{+} (T-\alpha_{L(T)},\, X_T-\overline{X}_T).$$

Since T is independent of X, using Lemma 3 we obtain

$$(T-\alpha_{L(T)},\, X_T-\overline{X}_T) \overset{\text{(d)}}{=} (\beta_{M(T)},\, \underline{X}_T).$$

□

Remark 1. In the right-hand side of distributional equality of Theorem 1 not only the two random elements are independent but they also have disjoint supports.

We are actually essentially finished. The rest is just straightforward calculations. But we push on.

3 The Wiener-Hopf factorization

One of our goals is to figure out the laws of $\overline{X}_t$ and $\underline{X}_t$ for $0 \leq t \leq \infty$. (The second $\leq$ is not a typo.) But observe that the law of $\overline{X}_T$ is a geometric sum of the laws of $\overline{X}_t$. In other words, the law of $\overline{X}_T$ specifies the generating function of $t \mapsto$ law of $\overline{X}_t$ and that is good enough. In our work below, I want you to keep in mind that $\overline{X}_T = X_{\alpha_{L(T)}}$ and $\underline{X}_T = X_{\beta_{M(T)}}$. Put it otherwise, you should keep in mind that the maximum and the minimum have been essentially expressed as sums of independent things.

3.1 From the first to the last time

The first order of business is to show how to see how to go

$$\text{from the law of } (\alpha, X_\alpha) \text{ to the law of } (\alpha_{L(T)}, X_{\alpha_{L(T)}}),$$

when T is an independent geometric$_0$ random variable. It pays to do that a little bit more generally.

Proposition 1 (From first to last)**.** *Let $S_n = (S_n^1, \ldots, S_n^d)$, $n \geq 0$, be a random walk in $\mathbb{R}^d$ Let $\mathfrak{t}$ be a stopping time with values in $\mathbb{N} \cup \{\infty\}$. Let $\mathfrak{t} = \mathfrak{t}_1 < \mathfrak{t}_2 < \cdots$ be the iterates of $\mathfrak{t}$ and $L(t)$ the counts: $L(t) = \sup\{k : \mathfrak{t}_k \leq t\}$. Let T be independent geometric$_0$ random variable: $\mathbb{P}(T = n) = q^n p$. Then, for $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$*

$$\mathbb{E} e^{i\theta \cdot S_{\mathfrak{t}_{L(T)}}} = \frac{1 - \mathbb{E} q^{\mathfrak{t}}}{1 - \mathbb{E} q^{\mathfrak{t}} e^{i\theta \cdot S_{\mathfrak{t}}}},$$

Proof. We just follow our nose: With $\mathfrak{t}_0 := 0$ we have

$$\begin{aligned}
\mathbb{E} e^{i\theta \cdot S_{\mathfrak{t}_{L(T)}}} &= \sum_{k \geq 0} \mathbb{E}[e^{i\theta \cdot S_{\mathfrak{t}_k}}; \mathfrak{t}_k \leq T < \mathfrak{t}_{k+1}] \\
&= \sum_{k \geq 0} \mathbb{E}[e^{i\theta \cdot S_{\mathfrak{t}_k}} (q^{\mathfrak{t}_k} - q^{\mathfrak{t}_{k+1}})] \\
&= \sum_{k \geq 0} \mathbb{E}[e^{i\theta \cdot S_{\mathfrak{t}_k}} q^{\mathfrak{t}_k} (1 - q^{\mathfrak{t}_{k+1} - \mathfrak{t}_k})] \\
&= \sum_{k \geq 0} \mathbb{E}(e^{i\theta \cdot S_{\mathfrak{t}_k}} q^{\mathfrak{t}_k}) \, \mathbb{E}(1 - q^{\mathfrak{t}_{k+1} - \mathfrak{t}_k})] \\
&= (1 - \mathbb{E} q^{\mathfrak{t}}) \sum_{k \geq 0} \big(\mathbb{E}(e^{i\theta \cdot S_{\mathfrak{t}}} q^{\mathfrak{t}})\big)^k = \frac{1 - \mathbb{E} q^{\mathfrak{t}}}{1 - \mathbb{E} q^{\mathfrak{t}} e^{i\theta \cdot S_{\mathfrak{t}}}}.
\end{aligned}$$

□

Caveat: The interpretation of $\mathbb{E} q^{\mathfrak{t}} e^{i\theta \cdot S_{\mathfrak{t}}}$ is $\mathbb{E}[q^{\mathfrak{t}} e^{i\theta \cdot S_{\mathfrak{t}}}; \mathfrak{t} < \infty]$. It is not necessary to include the indicator function in view of $|q| < 1$, but in an expression like $\mathbb{E} q^{\mathfrak{t}} e^{i\theta \cdot S_{\mathfrak{t}}}$ one ought to write $\mathbb{E}[e^{i\theta \cdot S_{\mathfrak{t}}}; \mathfrak{t} < \infty]$. Nevertheless, one does not, out of sheer laziness and under-the-rug conventions.

Applying the proposition to the random walk X_n, $n \geq 0$, with stopping time α and then with β, we have

$$\mathbb{E} e^{i\theta X_{\alpha_{L(T)}}} = \frac{1 - \mathbb{E} q^\alpha}{1 - \mathbb{E} q^\alpha e^{i\theta X_\alpha}}, \qquad \mathbb{E} e^{i\theta X_{\beta_{M(T)}}} = \frac{1 - \mathbb{E} q^\beta}{1 - \mathbb{E} q^\beta e^{i\theta X_\beta}}. \tag{9}$$

So we only need to figure out what the distributions of (α, X_α) and (β, X_β) are.

3.2 The laws at first times

Let F be the law of ξ_1 and let $\widehat{F}(\theta) = \mathbb{E} e^{i\theta \xi_1}$ be its characteristic function. Recall again that $X_{\alpha_{L(T)}} = \overline{X}_T$ and $X_{\beta_{M(T)}} = \underline{X}_T$. By Theorem 1,

$$X_T = \overline{X}_T \dot{+} \underline{X}_T.$$

We now take Fourier transforms of both sides. The left side is trivial: $\mathbb{E}e^{i\theta X_T} = \frac{1-q}{1-q\widehat{F}(\theta)}$. The right side is the product of the terms appearing in (9). Thus,

$$\frac{1-q}{1-q\widehat{F}(\theta)} = \frac{1-\mathbb{E}q^\alpha}{1-\mathbb{E}q^\alpha e^{i\theta X_\alpha}} \frac{1-\mathbb{E}q^\beta}{1-\mathbb{E}q^\beta e^{i\theta X_\beta}}.$$

On the other hand, by Theorem 1 again,

$$T \stackrel{\text{(d)}}{=} \alpha_{L(T)} \dot{+} \beta_{M(T)}$$

and so

$$\mathbb{E}z^T = \mathbb{E}z^{\alpha_{L(T)}}\, \mathbb{E}z^{\beta_{M(T)}}.$$

This gives

$$\frac{1-q}{1-qz} = \frac{1-\mathbb{E}q^\alpha}{1-\mathbb{E}(qz)^\alpha} \frac{1-\mathbb{E}q^\beta}{1-\mathbb{E}(qz)^\beta}.$$

and so

$$1-q = (1-\mathbb{E}q^\alpha)(1-\mathbb{E}q^\beta).$$

We thus arrive at

$$1-q\mathbb{E}e^{i\theta\xi_1} = \big(1-\mathbb{E}q^\alpha e^{i\theta X_\alpha}\big)\big(1-\mathbb{E}q^\beta e^{i\theta X_\beta}\big).$$

Taking logarithms and using the Taylor expansion

$$-\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}, \quad |x|<1,$$

we obtain

$$\sum \frac{1}{n}(q\mathbb{E}e^{i\theta\xi_1})^n = \sum \frac{1}{n}(\mathbb{E}q^\alpha e^{i\theta X_\alpha})^n + \sum_n \frac{1}{n}(\mathbb{E}q^\beta e^{i\theta X_\beta})^n$$

Use the following notation for the measures appearing on the right-hand side:

$$H_q(\cdot) = \mathbb{E}(q^\alpha, X_\alpha \in \cdot) = \mathbb{P}(\alpha \leq T, X_\alpha \in \cdot),$$
$$K_q(\cdot) = \mathbb{E}(q^\beta, X_\beta \in \cdot) = \mathbb{P}(\beta \leq T, X_\beta \in \cdot)$$

(keep in mind that $H_q(-\infty,0) = 0 = K_q[0,\infty)$) and let $\widehat{H}_q(\theta)$, $\widehat{K}_q(\theta)$ be their Fourier transforms. With this notation, we have

$$\sum \frac{1}{n} q^n \widehat{F}(\theta)^n = \sum \frac{1}{n}\widehat{H}_q(\theta)^n + \sum_n \frac{1}{n}\widehat{K}_q(\theta)^n.$$

Recalling that if μ is a bounded Borel measure with Fourier transform $\widehat{\mu}$ then $\widehat{\mu}^n$ is the Fourier transform of the convolution of μ a number n of times by itself, that is, if we let $\mu * \nu := \int_{\mathbb{R}} \mu(B-x)\nu(dx)$ and then let, inductively, $\mu^{*(n+1)} := \mu^n * \mu$, $n = 1, 2, \ldots$, with $\mu^{*1} := \mu$, we have $\widehat{\mu}^n = \widehat{\mu^{*n}}$. Therefore,

$$\sum \frac{1}{n} q^n F^{*n} = \sum \frac{1}{n} H_q^{*n} + \sum_n \frac{1}{n} K_q^{*n}.$$

This is an equality between Borel measures. Let B be a Borel subset of $[0,\infty)$. Since $K_q^{*n}[0,\infty)=0$, we have

$$\sum \frac{1}{n} H_q^{*n}(B) = \sum \frac{1}{n} q^n F^{*n}(B), \quad B \subset (0,\infty).$$

Similarly,

$$\sum \frac{1}{n} K_q^{*n}(B) = \sum \frac{1}{n} q^n F^{*n}(B), \quad B \subset (-\infty,0].$$

Taking Fourier transforms of both sides of the first display, we have

$$-\log(1-\widehat{H}_q(\theta)) = \sum_n \frac{1}{n} \widehat{H}_q(\theta)^n = \sum_n \frac{q^n}{n} \mathbb{E}(e^{i\theta X_n}; X_n > 0),$$

and similarly for the second display.

Proposition 2 (Baxter's equations)**.** *Let X_n, $n \ge 0$, be an arbitrary random walk in $\mathbb{R}$ with stationary and independent increments. Let α be such that $X_\alpha > 0$ for the first time (with $\alpha = \infty$ if the random walk remains ≤ 0 for ever), and let β be such that $X_\beta \le 0$ for the first time (with $\beta = \infty$ if $X_n > 0$ for all $n \ge 1$). Then, for $0 < q < 1$ and $\theta \in \mathbb{R}$,*

$$\mathbb{E}(q^\alpha e^{i\theta X_\alpha}) = 1 - \exp\big\{ -\sum \frac{q^n}{n} \mathbb{E}(e^{i\theta X_n}; X_n > 0)\big\}$$
$$\mathbb{E}(q^\beta e^{i\theta X_\beta}) = 1 - \exp\big\{ -\sum \frac{q^n}{n} \mathbb{E}(e^{i\theta X_n}; X_n \le 0)\big\}$$

3.3 Recurrence/transience properties of a general random walk

First, we have lots of corollaries to Baxter's equations.

Corollary 2.

$$\mathbb{E}q^\alpha = 1 - \exp\Big\{ -\sum_{n\ge1} \frac{q^n}{n} \mathbb{P}(X_n > 0)\Big\}, \qquad \mathbb{E}q^\beta = 1 - \exp\Big\{ -\sum_{n\ge1} \frac{q^n}{n} \mathbb{P}(X_n \le 0)\Big\},$$

Corollary 3.

$$\mathbb{P}(\alpha=\infty) = \exp\Big\{ -\sum_{n\ge1} \frac{1}{n} \mathbb{P}(X_n > 0)\Big\}, \qquad \mathbb{P}(\beta=\infty) = \exp\Big\{ -\sum_{n\ge1} \frac{1}{n} \mathbb{P}(X_n \le 0)\Big\}.$$

Proof. For $0<q<1$, we have $q^\alpha = q^\alpha \mathbf{1}_{\alpha<\infty}$. Hence $\lim_{q\uparrow 1} q^\alpha = \mathbf{1}_{\alpha<\infty}$. By monotone convergence, $\lim_{q\uparrow 1} \mathbb{E}q^\alpha = \mathbb{P}(\alpha<\infty)$. $\square$

Corollary 4. $\mathbb{P}(\alpha<\infty)=1 \iff \sum_{n\ge1}\frac{1}{n}\mathbb{P}(X_n>0)=\infty \iff \mathbb{P}(\overline{\lim}_{n\to\infty} X_n = \infty) = 1$. *In this case,*

$$\mathbb{E}\alpha = \exp \sum_{n\ge1} \frac{1}{n} \mathbb{P}(X_n > 0).$$

$\mathbb{P}(\beta<\infty)=1 \iff \sum_{n\ge1}\frac{1}{n}\mathbb{P}(X_n\le 0)=\infty \iff \mathbb{P}(\underline{\lim}_{n\to\infty} X_n = \infty) = 1$. *In this case,*

$$\mathbb{E}\beta = \exp \sum_{n\ge1} \frac{1}{n} \mathbb{P}(X_n \le 0).$$

Proof. If $\alpha < \infty$ a.s. then all iterates of α are finite a.s. This means that there are infinitely many records and so the random walk converges to infinity along a subsequence. The converse also holds. Hence $\mathbb{P}(\alpha < \infty) = 1$ is equivalent to $\mathbb{P}(\overline{\lim}_{n\to\infty} X_n = \infty) = 1$. The middle equivalence is obtained from the expression for $\mathbb{P}(\alpha = \infty)$ which is zero if and only if the exponent $\sum_{n\geq 1} \frac{1}{n}\mathbb{P}(X_n > 0)$ is infinity. □

Corollary 5. *We always have*

$$\mathbb{E}\alpha = \frac{1}{\mathbb{P}(\beta = \infty)}, \quad \mathbb{E}\beta = \frac{1}{\mathbb{P}(\alpha = \infty)}.$$

Corollary 6. *Cases I, II and III below are mutually exclusive:*

	$\mathbb{P}(\alpha < \infty)$	$\mathbb{E}\alpha$	$\mathbb{P}(\beta < \infty)$	$\mathbb{E}\beta$	$\overline{\lim}\, X_n$	$\underline{\lim}\, X_n$	$\sup_n X_n$	$\inf_n X_n$
I	1	∞	1	∞	$+\infty$	$-\infty$	$+\infty$	$-\infty$
II	1	$< \infty$	< 1	∞	$+\infty$	$+\infty$	$+\infty$	$> -\infty$
III	< 1	∞	1	$< \infty$	$-\infty$	$-\infty$	$< +\infty$	$-\infty$

(The last 4 *columns should be interpreted almost surely.)*

Proof. We classify cases according to whether $\mathbb{P}(\alpha < \infty)$, $\mathbb{P}(\beta < \infty)$ are $= 1$ or < 1 (4 cases). But the case where they both equal to 1 is impossible, owing to the formulae of Corollary 5. The equivalences on each line follow from earlier arguments. □

Terminology: We say that the random walk is recurrent if *I* holds. Otherwise, we say that it is transient. If *II* holds we say that it escapes to $+\infty$. If *III* holds we say that it escapes to $-\infty$.

Proposition 3. *If $\mathbb{E}\xi_1$ exists then, with reference to the previous corollary, I $\iff \mathbb{E}\xi_1 = 0$, II $\iff \mathbb{E}\xi_1 > 0$, III $\iff \mathbb{E}\xi_1 < 0$.*

Proof. If $\mathbb{E}\xi_1 > 0$, then $X_n \to \infty$ a.s., and so this is case *II*. Similarly for case *III*. The only remaining possibility is $\mathbb{E}\xi_1 = 0$, and this is case *I*. □

3.4 Overall extrema: Spitzer's identity

Spitzer's identity gives the (Fourier transform of the) distribution of $\overline{X}_T$, the maximum up to a random memoryless time:

Theorem 2 (Spitzer's identity)**.** *Let X_n, $n \geq 0$, be any random walk in $\mathbb{R}$ with i.i.d. increments and T an independent geometric$_0$ time: $\mathbb{P}(T = n) = q^n p$. Then*

$$\frac{1}{p}\,\mathbb{E}e^{i\theta\overline{X}_T} = \sum_{n=0}^{\infty} q^n \mathbb{E}e^{i\theta X_n} = \exp\sum_{n\geq 1} \frac{q^n}{n}\mathbb{E}e^{i\theta X_n^+}.$$

Proof. The first of (9) gives

$$\mathbb{E}e^{i\theta\overline{X}_T} = \frac{1 - \mathbb{E}q^\alpha}{1 - \mathbb{E}q^\alpha e^{i\theta X_\alpha}}$$

The terms on the RHS of this identity are expressed by Baxter's equation and Corollary 2:

$$
\begin{aligned}
\mathbb{E}e^{i\theta\overline{X}_T} &= \exp\Big\{-\sum_{n\ge1}\frac{q^n}{n}\mathbb{P}(X_n>0)+\sum_{n\ge1}\frac{q^n}{n}\mathbb{E}(e^{i\theta X_n};X_n>0)\Big\}\\
&= \exp\Big\{\mathbb{E}\sum_{n\ge1}\frac{q^n}{n}(e^{i\theta X_n}\mathbf{1}_{X_n>0}-\mathbf{1}_{X_n>0})\Big\}\\
&= \exp\Big\{\mathbb{E}\sum_{n\ge1}\frac{q^n}{n}(e^{i\theta X_n}\mathbf{1}_{X_n>0}+\mathbf{1}_{X_n=0}-1)\Big\}\\
&= \exp\Big\{\sum_{n\ge1}\frac{q^n}{n}(\mathbb{E}e^{i\theta X_n^+}-1)\Big\} \qquad (10)\\
&= (1-q)\,\exp\Big\{\sum_{n\ge1}\frac{q^n}{n}\mathbb{E}e^{i\theta X_n^+}\Big\}.
\end{aligned}
$$

□

Define

$$\overline{X}_\infty=\sup_n X_n=\lim_{n\to\infty}\overline{X}_n.$$

Corollary 7. $\mathbb{P}(\overline{X}_\infty<\infty)=1 \iff \sum_{n\ge1}\frac{1}{n}\mathbb{P}(X_n>0)<\infty.$

Theorem 3. *If* $\mathbb{P}(\overline{X}_\infty<\infty)=1$ *we have*

$$\mathbb{E}e^{i\theta\overline{X}_\infty}=\exp\sum_{n\ge1}\frac{1}{n}(\mathbb{E}e^{i\theta X_n^+}-1).$$

Proof. From Spitzer's identity, in the form of eq. (10), letting $q\uparrow1$ we have $T\uparrow\infty$ a.s. and so $\overline{X}_T\uparrow\overline{X}_\infty$ a.s. Since $\mathbb{P}(\overline{X}_\infty<\infty)=1$ we have $\sum_n\frac{1}{n}\mathbb{P}(X_n>0)<\infty$. Using this and the dominated convergence theorem on the RHS of (10) we have that its limit equals $\exp\sum_{n\ge1}\frac{1}{n}(\mathbb{E}e^{i\theta X_n^+}-1)$. □

4 The difficulties in continuous time

If I is an interval, a function $f:I\to\mathbb{R}$ is said to have at most discontinuities of the first kind if, for all $t\in I$, the limits $\lim_{\varepsilon\downarrow0}f(t\pm\varepsilon)$ exist in $\mathbb{R}$. It is said to be càdlàg if it has at most discontinuities of the first kind and if it is right continuous. We let $D(I)$ be the set of càdlàg functions on I. We say that a collection of random variables X_t, $t\ge0$, on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ is Lévy if it has stationary and independent increments and if, for all ω, the path $t\mapsto X_t(\omega)$ is càdlàg.

To appreciate the difficulties arising in trying to study the maxima of a Lévy process, let us consider a Brownian motion W_t, $t\ge0$, and its minimum process

$$\underline{W}_t=\inf_{0\le s\le t}W_s=\min_{0\le s\le t}W_s.$$

A Brownian motion is a Lévy process with continuous paths. It follows (by a careful application of the central limit theorem for triangular arrays) that a Brownian motion must have normal finite-dimensional distributions. It is also known that $Q_t = W_t - \underline{W}_t$ is a strong Markov process with the property that

$$\inf\{t > 0 : Q_t = 0\} = 0 = \inf\{t > 0 : Q_t > 0\} \text{ a.s.}$$

This forces the closed set $\{t \geq 0 : Q_t = 0\}$ to be perfect: it contains all its limit points; moreover it is nowhere dense. Such a set is uncountable and has Lebesgue measure 0. Hence we cannot talk of the first time $t > s$ such that $W_t \leq \underline{W}_s$ since this first time is equal to s. And yet there are such times, in fact there are uncountably many such times. If we are to follow an approach similar to that in discrete time, we must find a way to enumerate those times. Since we cannot, we will try to find a way to "count" those times, that is, to count the "number" of occurrences of these times on an interval. Recall, from discrete-time theory, that $L(t)$ counts the iterates of α up to time t; conversely, knowledge of $L(t)$ implies knowledge of the iterates of α. It turns out that the converse is possible for a Brownian motion and, more generally, for a Lévy process. What we need to do is find an increasing random function $L(t)$, $t \geq 0$, such that its point of increase (that is, the support of the random Borel measure defined by L) coincides with the set

$$\mathfrak{Z} = \{t \geq 0 : W_t = \underline{W}_t\} = \{t \geq 0 : Q_t = 0\}.$$

We shall then define the inverse of the function L and think of it as a proxy for the iterates of α. The definition of L can be done in many ways, for example, using stochastic calculus. But we shall follow a probabilistic approach.

The crux of the matter lies on the fact that every open subset of $\mathbb{R}$ is written uniquely as the countable union of open intervals. Applying this to $\mathfrak{Z}$, we write $\mathfrak{Z}$ as countable union of intervals. If I is one of these open intervals then $W_t > 0$ for all $t \in I$, but $W_s = 0$ if s is one of the endpoints of I. So, even though we cannot enumerate $\mathfrak{Z}$ we can enumerate the intervals of its complement and this will form the basis of the construction of $L(t)$.

5 The reflected Lévy process as a Markov process

Let X_t, $t \geq 0$, be a general Lévy process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For our purposes, we may take $\Omega = D[0, \infty)$, and $\mathscr{F}$ the smallest σ-algebra containing all open (in the Skorokhod J_1-topology) subsets of $D[0, \infty)$. Let $\mathscr{N}$ be the class of all subsets (the null sets) of Ω such that there is $A \in \mathscr{F}$ with $N \subset A$ and $\mathbb{P}(A) = 0$. Let $\mathscr{F}_t := \sigma(X_0, 0 \leq s \leq t) \vee \mathscr{N}$, $t \geq 0$, and let $\mathscr{F}_\infty := \mathscr{F} \vee \mathscr{N}$. (The operation of enlarging a σ-algebra by the null sets of a possibly bigger σ-algebra is called completion.) Then $\mathscr{F}_t$, $t \geq 0$, is a filtration (this means that $t_1 < t_2 \Rightarrow \mathscr{F}_{t_1} \subset \mathscr{F}_{t_2}$) with respect to which X is adapted (this means that X_t is measurable with respect to $\mathscr{F}_t$ for all t). We say that the filtration $\mathscr{F}_t$, $t \geq 0$, is complete (because it contains the null subsets of Ω). What is less obvious is that the operation of completion has done something that should be thought of as magic: it has made the filtration right-continuous, that is,

$$\mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{\varepsilon > 0} \mathscr{F}_{t+\varepsilon}.$$

This is not a trivial result. I cannot overemphasize its importance however. Indeed, from this and the independent increment property we obtain that

$$A \subset \mathscr{F}_0 \Rightarrow \mathbb{P}(A) \in \{0, 1\}. \tag{11}$$

We say that $\mathscr{F}_0 = \mathscr{F}_{0+}$ is trivial. But it is far from containing nothing. It contains lots of interesting sets and measures lots of interesting random variables. The two random variables that are of interest are

$$\mathfrak{r} := \inf\{t > 0 : X_t = \inf_{0 \le s \le t} X_s\}.$$

This is the first *return* to 0 of the reflected process Q. It plays the role of β from our discrete-time theory. But, since $\mathfrak{r}$ is measurable with respect to $\mathscr{F}_{0+}$ and since (11) holds we immediately get

$$\mathbb{P}(\mathfrak{r} = 0) \in \{0, 1\}.$$

The second random variable of interest is

$$\mathfrak{s} := \inf\{t > 0 : X_t > \inf_{0 \le s \le t} X_s\}.$$

Since, for $t < \mathfrak{s}$ we have $X_t = \inf_{0 \le s \le t} X_s$, the time $\mathfrak{s}$ is the *sojourn* time of Q at 0 (the amount of time that Q spends at 0, starting from $Q_0 = 0$). Again, $\mathfrak{s}$ is measurable with respect to $\mathscr{F}_{0+}$ and so

$$\mathbb{P}(\mathfrak{s} = 0) \in \{0, 1\}.$$

We thus have four cases: $\mathbb{P}(\mathfrak{r} = \mathfrak{s} = 0) = 1$ or $\mathbb{P}(\mathfrak{r} = 0, \mathfrak{s} > 0) = 1$ or $\mathbb{P}(\mathfrak{r} > 0, \mathfrak{s} = 0) = 1$ or $\mathbb{P}(\mathfrak{r} > 0, \mathfrak{s} > 0) = 1$. But, since $\mathfrak{s} > 0 \Rightarrow \mathfrak{r} = 0$, the fourth cases is not possible. Hence there are, really, only three cases:

Case 1. $\mathbb{P}(\mathfrak{r} = \mathfrak{s} = 0) = 1$: almost surely, every path of Q leaves 0 immediately and returns to 0 immediately.

Case 2. $\mathbb{P}(\mathfrak{s} > 0) = 1$ (and hence $\mathbb{P}(\mathfrak{r} = 0) = 1$): almost surely, every path of Q starts from 0 and stays at 0 for a while.

Case 3. $\mathbb{P}(\mathfrak{r} > 0) = 1$ (and hence $\mathbb{P}(\mathfrak{s} = 0) = 1$): almost surely, every path of Q leaves 0 immediately and returns to 0 at a strictly positive time.

$$\boxed{\textbf{Another picture worth a hundred words goes here}} \tag{12}$$

Let us take a closer look at the reflected process. We shall show that it is a strong Markov process. This means that the law of the future of Q after some time s, conditional on its past, depends only on Q_s. We can see this explicitly using A little algebra of the max-plus type gives, for $s < t$,

$$\begin{aligned}
Q_t &= X_t - \inf_{0 \le u \le t} X_u \\
&= \sup_{0 \le u \le t} (X_t - X_u) \\
&= \sup_{s \le u \le t} (X_t - X_u) \vee \sup_{0 \le u \le s} (X_t - X_u) \\
&= \sup_{s \le u \le t} (X_t - X_u) \vee \sup_{0 \le u \le s} (X_t - X_s + X_s - X_u) \\
&= \sup_{s \le u \le t} (X_t - X_u) \vee \big(\sup_{0 \le u \le s} (X_s - X_u) + X_t - X_s \big) \\
&= \sup_{s \le u \le t} (X_t - X_u) \vee \big(Q_s + X_t - X_s \big).
\end{aligned}$$

Hence Q_t is a function of Q_s and $(X_t - X_s, t > s)$. But Q_s is measurable with respect to $\mathscr{F}_s$, while $(X_t - X_s, t > s)$ is independent of $\mathscr{F}_s$. Whence the Markov property. Repeating the argument for a stopping time $\mathfrak{t}$ with respect to $\mathscr{F}.$ rather than t we obtain the strong Markov property. Notice

$$Q_t = \sup_{0 \le u \le t} (X_t - X_u) \vee \big(Q_0 + X_t\big).$$

So if, for $x \in \mathbb{R}$, we let $\mathbb{P}_x$ be the law of $x + X.$ then, for $x \ge 0$, we can think of $\mathbb{P}_x$ as the law of $Q.$ starting from $Q_0 = x$. The strong Markov property then reads

$$\mathbb{P}(Q_{\mathfrak{t}+\cdot} \in \cdot | \mathscr{F}_{\mathfrak{t}}) = \mathbb{P}_{Q_{\mathfrak{t}}}(Q. \in \cdot) \text{ a.s. on } \{\mathfrak{t} < \infty\}.$$

The usual continuity with respect to the initial state (if y is close to x then the $\mathbb{P}_y$-law of Q_t is close to the $\mathbb{P}_x$-law of Q_t, for each t, is obvious from the formula above giving the explicit dependence on the initial state).

Introduce next a shift operator θ_t on the paths of X that takes $X\cdot$ and gives $X_{t+\cdot}$. (Note that this shift is *substantially* different from the shift we used in discrete time because there we shifted increments rather than positions.) So if Y is a random variable then $Y \circ \theta_t$ is the random variable at the shifted path: the random variable as seen when we shift the origin of time from 0 to t. For example, if T_A is the infimum of all $s > 0$ such that $X_t \in A$ then $T_A \circ \theta_t$ is the infimum of all $s > 0$ such that $X_{t+s} \in A$. We may also define $\theta_{\mathfrak{t}}$ for any finite random variable (in particular a stopping time) that takes values in the set of times. The strong Markov property can thus be equivalently written as

$$\mathbb{E}(Y \circ \theta_{\mathfrak{t}} | \mathscr{F}_{\mathfrak{t}}) = \mathbb{E}_{Q_{\mathfrak{t}}} Y \text{ a.s. on } \{\mathfrak{t} < \infty\}.$$

for any bounded measurable function Y.

Note that $\mathfrak{Z}$ contains lots of stopping times. For instance, we may define the first return of Q to 0 after the first time that Q has exceeded a given level u and has spent time at least v above level u. If we call $\mathfrak{t}$ this rather complicated stopping time then the strong Markov property at $\mathfrak{t}$ tells us that the path of the process in the right neighborhood of $\mathfrak{t}$ behaves like in one of the above 3 cases. This gives a further picture of what the paths of Q are like.

Remark 2. In what follows we construct the so-called local time of $Q_t = X_t - \underline{X}_t$ at 0. The only essential property of Q required for this construction is that it is a strong Markov process. Hence the construction is much more general.

6 Excursions from the origin (moderato)

We are mainly interested in the reflected process $Q_t = X_t - \underline{X}_t$, $t \ge 0$, that was shown to be strong Markov. The development below is much more general. So we shall be thinking of Q as a strong Markov process in a nice space (e.g. $\mathbb{R}^d$) relative to a right-continuous complete filtration with right-continuous sample paths and with a certain continuity condition with respect to the initial state (Feller process).

An *excursion* of Q away from 0 is, by definition, a piece

$$Q_t,\ g < t < d,$$

of the process such that $Q_t > 0$ for all $t \in (g, d)$, and the interval (g, d) is maximal; in the sense that if we enlarge it we shall find a zero of Q or a point that can be written as limits of zeros of Q. The interval (g, d) is called an excursion interval and it may be the case that d is at ∞.

Lemma 4. *If U is a nonempty open subset of $\mathbb{R}$ then there is a unique pairwise disjoint family $\mathscr{I}$ of open intervals such that*

$$U = \biguplus_{I \in \mathscr{I}} I.$$

The family $\mathscr{I}$ is countable. The endpoints of each $I \in \mathscr{I}$ are not in U.

Proof. Exercise. $\square$

The members of $\mathscr{I}$ are called components of U. This is justified from the fact that if U is considered as a topological space with the embedded topology then the $I \in \mathscr{I}$ are the connected components of U, that is, those subsets of U that are simultaneously open and closed in U.

Consider again the random set $\mathfrak{Z}$ of zeros of Q. The strong Markov property of Q implies that Q is a *regenerative process* over $\mathfrak{Z}$ (some people say that the set $\mathfrak{Z}$ is a *regenerative set* for Q):

$$\mathbb{P}(Q_{\mathfrak{t}+\cdot} \in \cdot | \mathscr{F}_{\mathfrak{t}}) = \mathbb{P}(Q_\cdot \in \cdot) \text{ a.s. on } \underbrace{\mathfrak{t} \in \mathfrak{Z}}_{\mathfrak{t}<\infty,\, Q_{\mathfrak{t}}=0}.$$

Applying the lemma above to the set $\overline{\mathfrak{Z}}$ we find that there exists a unique countable collection $\mathscr{I}$ of pairwise disjoint open intervals such that $\overline{\mathfrak{Z}}^c = [0, \infty) - \overline{\mathfrak{Z}} = \bigoplus_{I \in \mathscr{I}} I$. The right-continuity of X (and hence of Q) implies that every point in $\overline{\mathfrak{Z}} - \mathfrak{Z}$ is isolated from the right. If $I = (g, d)$ denotes the typical element of $\mathscr{I}$ then $\mathfrak{Z}^c$ is a countable union of intervals of the form (g, d) or $[g, d)$.

What we need to do next is find a nontrivial random locally finite Borel measure L whose support is $\mathfrak{Z}$ that is compatible with the regenerative property. It will turn out that this L is unique up to multiplicative constant. If $\mathfrak{Z}$ has positive Lebesgue measure (Case 2) then we can take L to be the restriction of the Lebesgue measure on $\mathfrak{Z}$. If $\mathfrak{Z}$ is countable (Case 3) then we can take L to be proportional to the cardinality measure of $\mathfrak{Z}$, i.e. $B \mapsto \sum_{z \in \mathfrak{Z} \cap B} \delta_z$ or some variant theoreof. The only nontrivial case is Case 1. As we shall see, in this case, $\mathfrak{Z}$ is topologically just as the set of zeros of a Brownian motion, that is, it is uncountable, it is nowhere dense and it has Lebesgue measure 0. For such, perfect, sets, finding nontrivial measures on them is a nontrivial matter.

7 Counting and indexing excursions in Case 1

We now properly address the following problems: (i) How do we count how many excursions we have on an interval of time $[0, t]$? (ii) How do we index these excursions properly?

Regarding (i), we will replace the verb "to count" by the verb "to measure" because there are infinitely (albeit countably many) excursions on every interval. We shall thus construct an increasing function $L(t)$, $t \geq 0$, that increases precisely when an excursion occurs. The Borel measure defined by L will be a proper replacement of the counting one. Regarding (ii), and having constructed L, we will simply consider the generalized inverse function L_ν^{-1}, $\nu \geq 0$, which has countably many points of discontinuity: when ν is a point of discontinuity of it, there is an excursion indexed by ν. This is the correct point of view. Once these processes are established and understood, we're ready to breeze through.[3]

7.1 Counting excursions

We shall consider Case 1 (the difficult case) here, that is, $\mathfrak{r} = \mathfrak{s} = 0$ a.s. Since we cannot talk about the first excursion (interval), we must find a way to consistently enumerate the excursions (after all, there are countably many of them).

For $u > 0$ we say that an excursion is of type u or that it is a u-excursion if it has duration at least u. Since there can be at most finitely many such excursions over any compact interval, we can enumerate them in a manner compatible to the ordering of real numbers. If we let $\mathscr{I}(u)$ be the collection of u-excursions, we can write

$$\mathscr{I}(u) = \{(g_1(u), d_1(u)),\ (g_2(u), d_2(u)), \ldots\},$$

where

$$0 < g_1(u) < d_1(u) < g_2(u) < d_2(u) < \cdots$$

Note that $\mathscr{I}(u)$ may actually be a finite set, in which case there is a last u-excursion of infinite duration.

It is not difficult to see that

Lemma 5. *There is some $u > 0$ such that $\mathscr{I}(u) \neq \varnothing$ a.s.*

The point is that, whereas every path has a u-excursion for some $u > 0$, there is a u-excursion for a u that does not depend on the path, almost surely for all paths. We shall use the letter c_0 for such a u and call it the "gauge constant".

Notice that $\mathscr{I}(u)$ increases as u decreases, so $\mathscr{I} = \bigcup_{u>0} \mathscr{I}(u) = \bigcup_{n\geq 1} \mathscr{I}(1/n)$. Define next the random variable

$$I_1(u) = (g_1(u), d_1(u)), \quad \ell_1(u) = g_1(u) - d_1(u).$$

Notice that the intervals in $\mathscr{I}$ are ordered in a manner compatible with the ordering of real numbers. So if $I, J \in \mathscr{I}$ we can say "I before J" if the right endpoint of I is smaller than the left endpoint of J.

Lemma 6. *Let $0 < u < v$. Then $I_1(v)$ cannot be before $I_1(u)$. Furthermore,*
(i) $I_1(u)$ before $I_1(v) \iff \ell_1(u) \leq v$
(ii) $I_1(u) = I_1(v) \iff \ell_1(u) > v$.

We shall also need

Lemma 7. *The random variables $d_k(u)$, $k = 1, 2, \ldots$, $u > 0$, are all stopping times. (But the $g_k(u)$, $k = 1, 2, \ldots$, $u > 0$, are not.)*

A remarkable thing now happens: the function $(x, y) \mapsto \mathbb{P}(\ell_1(x) > y)$ is multiplicative in the following sense. (Keep in mind that $\mathbb{P}(\ell_1(x) > y) = 1$ for $y \leq x$.)

[3] breeze through something: to do something very easily or confidently

Lemma 8. *For* $0 < x < y < z$,

$$\mathbb{P}(\ell_1(x) > z) = \mathbb{P}(\ell_1(x) > y)\,\mathbb{P}(\ell_1(y) > z).$$

Proof. We have

$$\begin{aligned}\{I_1(x) = I_1(z)\} &= \{I_1(x) = I_1(y) = I_1(z)\} \\ &= \{I_1(y) = I_1(z)\} \setminus \{I_1(x) \text{ before } I_1(y),\, I_1(y) = I_1(z)\}\end{aligned}$$

so

$$\mathbb{P}(I_1(x) = I_1(z)) = \mathbb{P}(I_1(y) = I_1(z)) - \mathbb{P}(I_1(x) \text{ before } I_1(y),\, I_1(y) = I_1(z)).$$

Notice that $\{I_1(x) \text{ before } I_1(y)\} \in \mathscr{F}_{d_1(x)}$ because by the end of the first x-excursion we can tell decide whether its duration was larger than y or not.

$$\mathbb{P}(I_1(x) \text{ before } I_1(y),\, I_1(y) = I_1(z) | \mathscr{F}_{d_1(x)}) = \mathbf{1}_{I_1(x) \text{ before } I_1(y)}\, \mathbb{P}(I_1(y) = I_1(z) | \mathscr{F}_{d_1(x)}).$$

But, if $I_1(x)$ before $I_1(y)$ then $I_1(y+\varepsilon)\circ\theta_{d_1(x)} = I_1(y+\varepsilon)$ for any $\varepsilon > 0$. (In plain words: If the first x-excursion is before the first y excursion then the first $(y+\varepsilon)$-excursion is after the first $(y+\varepsilon)$-excursion after the first x-excursion.) So, by the strong Markov property, and the fact that $Q_{d_1(x)} = 0$,

$$\begin{aligned}\mathbb{P}(I_1(y) = I_1(z) | \mathscr{F}_{d_1(x)}) &= \mathbb{P}(I_1(y)\circ\theta_{d_1(x)} = I_1(z)\circ\theta_{d_1(x)} | \mathscr{F}_{d_1(x)}) \\ &= \mathbb{P}_{Q_{d_1(x)}}(I_1(y) = I_1(z)) = \mathbb{P}(I_1(y) = I_1(z))\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{P}(I_1(x) = I_1(z)) &= \mathbb{P}(I_1(y) = I_1(z))\,[1 - \mathbb{P}(I_1(x) \text{ before } I_1(y))] \\ &= \mathbb{P}(I_1(y) = I_1(z))\,\mathbb{P}(I_1(x) = I_1(y)).\end{aligned}$$

We now apply Lemma 6. □

Remark 3 (additivity)**.** Note that if a function $f : [0, \infty) \to \mathbb{R}$ satisfies

$$f(x, z) = f(x, y) + f(y, z), \text{ for all } x < y < z$$

and $f(x, x) = 0$ then taking any $c_0 > 0$ and defining $g(x) = f(x, c_0)\mathbf{1}_{x \le c_0} - f(c_0, x)\mathbf{1}_{x > c_0}$ we have that $f(x, y) = g(x) - g(y)$.

Applying the above remark to the function $-\log \mathbb{P}(\ell_1(x) > y)$ we see that

$$\mathbb{P}(\ell_1(x) > y) = \frac{\Psi(y)}{\Psi(x)}, \quad \Psi(x) := \begin{cases} 1/\mathbb{P}(\ell_1(x) > c_0), & x \le c_0 \\ \mathbb{P}(\ell_1(c_0) > x), & x \ge c_0 \end{cases}, \quad \Psi(c_0) = 1.$$

Note that $\Psi(x) < \infty$ for all $x > 0$; and that $x < y \Rightarrow \Psi(y) \ge \Psi(x)$; so define $\Psi(\infty) = \lim_{x\to\infty} \Psi(x)$. Also note that Ψ is right-continuous. Moreover, precisely because we are in Case 1, we have $\lim_{x\downarrow 0} \Psi(x) = \infty$. Note that $\Psi(\infty)$ could be 0 or positive. If $\Psi(\infty) > 0$ then there is a positive probability that the first c_0-excursion have infinite length and so there is an infinite-length excursion a.s. This means that Q never returns to 0 after a while, so $\mathfrak{Z}$ is a bounded set. If $\Psi(\infty) = 0$ then there are infinitely many c_0 excursions a.s., and so $\mathfrak{Z}$ is unbounded. Hence $\mathbb{P}(\mathfrak{Z}$ is bounded) has probability 1 or 0 according as $\Psi(\infty) > 0$ or not. These remarks can be turned into recurrence properties for the Lévy process. Note that if $\mathfrak{Z}$ is bounded then $\varliminf_{t\to\infty} X_t > -\infty$.

Define now

$$N_a(t) := \sup\{n \geq 1 : g_n(a) < t\}, \quad a > 0,\, t > 0.$$

Note that $N_x(g_1(x)) = 0$ and $N_x(g_1(x-\varepsilon)) = 0$ for $\varepsilon > 0$ because $I_1(x-\varepsilon)$ is either before or the same as $I_1(x)$. Hence $N_x(g_1(y))$ is nontrivial only when $x \leq y$; moreover, $N_x(d_1(y)) = N_x(g_1(y)) + 1$.

Theorem 4 (existence of continuous local time)**.** *For all $t > 0$, the $\lim_{a\downarrow 0} N_a(t)/\Psi(a)$ exists a.s. and is denoted by $L(t)$. The random function $t \mapsto L(t)$ is defined a.s., it is increasing, it is continuous and the support of the Borel measure defined by L is $\overline{\mathfrak{Z}}$.*

This theorem will be proved in several steps. First, we shall define $L(t)$ for t being a right endpoint of an excursion interval. To do this, we fix a $u > 0$, pick a u-excursion and loot at its right endpoint. We first pick the first u-excursion, defined over $(g_1(u), d_1(u))$ and look at $L(d_1(u))$. Since we claim that this equals $\lim_{a\downarrow 0} N_a(d_1(u))/\Psi(a)$, we need to prove that this limit exists. This is not particularly hard because we can explicitly compute the distribution of $N_a(d_1(u))$ and then use a martingale argument.

Lemma 9. *For $x < y$ such that $\Psi(x) > 0$, the random variable $N_x(d_1(y))$ is geometric with parameter $\Psi(y)/\Psi(x)$, i.e.*

$$\mathbb{P}(N_x(d_1(y)) = k) = \left(1 - \frac{\Psi(y)}{\Psi(x)}\right)^{k-1} \frac{\Psi(y)}{\Psi(x)}, \quad k = 1, 2, \ldots$$

Moreover, $N_x(d_1(y))$ is independent of $Q_{g_1(y)+t}$, $t \geq 0$.

Sketch of proof. We may as well prove that $N_x(g_1(y))$ is geometric$_0$ with parameter $\Psi(y)/\Psi(x)$. We only show that $\mathbb{P}(N_x(g_1(y)) = 0) = \Psi(y)/\Psi(x)$ and leave the rest as an exercise for the reader. We have

$$\mathbb{P}(N_x(g_1(y)) = 0) = \mathbb{P}(I_1(x) = I_1(y)) = \mathbb{P}(\ell_1(x) > y) = \frac{\Psi(y)}{\Psi(x)}.$$

□

Lemma 10. *Fix u such that $\Psi(u) > 0$. Then*

$$\mathcal{L}(d_1(u)) := \lim_{a\downarrow 0} \frac{N_a(d_1(u))}{\Psi(a)} \text{ exists a.s. and in } L^1.$$

Moreover, $\mathcal{L}(d_1(u))$ is independent of $I_1(u)$ and $\mathbb{P}(\mathcal{L}(d_1(u)) > t) = e^{-\Psi(u)t}$, $t \geq 0$.

Proof. It is easy to see that the limit in distribution exists. Indeed, since $\Psi(a) \uparrow \infty$ as $a \downarrow 0$,

$$\lim_{a\downarrow 0} \mathbb{P}\left(\frac{N_a(d_1(u))}{\Psi(a)} > t\right) = \lim_{a\downarrow 0}\left(1 - \frac{\Psi(u)}{\Psi(a)}\right)^{t\Psi(a)} = e^{-\Psi(u)t}.$$

Since $N_a(d_1(u))$ is independent of $Q_{g_1(u)+\cdot}$, it follows that $\overline{\lim}_{a\downarrow 0} N_a(d_1(u))/\Psi(a)$ is independent of $I_1(u)$. It remains to show that the a.s. limit actually exists. Define $\mathscr{G}_a := \sigma(I_1(a), I_2(a), \dots, I_{N_a(d_1(u))})$, $0 < a < u$. Note that $x < y \Rightarrow \mathscr{G}_x \supset \mathscr{G}_y$. Take $x < y < u$ and show that

$$\mathbb{E}\left[\frac{N_x(d_1(u))}{\Psi(x)}\middle|\mathscr{G}_y\right] = \frac{N_y(d_1(u))}{\Psi(y)}.$$

This is because $N_x(d_1(y))$ is a geometric random variable with parameter $\Psi(y)/\Psi(x)$, so $\mathbb{E}N_x(d_1(y)) = \Psi(x)/\Psi(y)$. So $N_x(d_1(u))$, conditional on $\mathscr{G}_y$ is the sum of $N_y(d_1(u))$ independent such geometric random variables. So $\mathbb{E}[N_x(d_1(u))|\mathscr{G}_y] = N_y(d_1(u))\Psi(x)/\Psi(y)$. And so we have a reverse martingale. From this, convergence a.s. and in L^1 follows. $\square$

Proof of Theorem 4. We define $\mathcal{L}(d_1(u))$ to be equal to $\lim_{a\downarrow 0} \frac{N_a(d_1(u))}{\Psi(a)}$. By applying the strong Markov property at the right endpoints of u-excursions we have that, for all k, $\lim_{a\downarrow 0} \frac{N_a(d_k(u))}{\Psi(a)}$ exists. We denote this limit by $\mathcal{L}(d_k(u))$. By the strong Markov property again we have that $\mathcal{L}(d_1(u))$, $\mathcal{L}(d_2(u)) - \mathcal{L}(d_1(u))$, $\mathcal{L}(d_3(u)) - \mathcal{L}(d_2(u)), \dots$ are independent exponential random variables. Since, decreasing u only increases the set of u-excursions, we have defined $\mathcal{L}(t)$ for all t that are endpoints of some excursion. Let $\mathfrak{D}$ be the set of these endpoints. Clearly, $\mathcal{L}$ is increasing on $\mathfrak{D}$. We then define

$$L(s) := \begin{cases} \mathcal{L}(s), & \text{if } s \in \mathfrak{D} \\ \inf_{t\in\mathfrak{D}, t>s} \mathcal{L}(t), & \text{otherwise.} \end{cases}$$

All we need to show is that $\inf_{t\in\mathfrak{D},t>s} = \sup_{t\in\mathfrak{D},t<s}$ is $s \notin \mathfrak{D}$, that is, that $L(\mathfrak{D})$ is dense in $[0,\infty)$. If it is not dense in $[0, d_1(c_0)]$ then there exists $\varepsilon > 0$ and $t_1, t_2 \in \mathfrak{D}$, $t_1 < t_2$, such that $L(t_2) - L(t_1) > \varepsilon$. Conditioning on the geometric random variable $N_a(d_1(c_0))$ we have

$$\begin{aligned}
&\mathbb{P}(L(d_k(a)) - L(d_{k-1}(a)) \le \varepsilon, 1 \le k \le N_a(d_1(c_0))) \\
&\quad= \sum_{n=1}^{\infty} \mathbb{P}(N_a(d_1(c_0)) = n)\, \mathbb{P}(L(d_k(a)) - L(d_{k-1}(a)) \le \varepsilon, 1 \le k \le n | N_a(d_1(c_0)) = n) \\
&\qquad= \sum_{n=1}^{\infty} \left(1 - \frac{\Psi(c_0)}{\Psi(a)}\right)^{n-1} \frac{\Psi(c_0)}{\Psi(a)} \left(1 - e^{-\Psi(a)\varepsilon}\right)^n = \frac{\Psi(c_0)(1 - e^{-\Psi(a)\varepsilon})}{\Psi(c_0) + (\Psi(a) - \Psi(c_0))e^{-\Psi(a)\varepsilon}}
\end{aligned}$$

and this converges to 1 as $a \to 0$. Since, for all $\varepsilon > 0$, there is no jump of L bigger than ε it follows that L has no jumps on $[0, d_1(c_0)]$. Using the strong Markov property we see that L has no jumps anywhere.
By the construction of L, if t is a right point of increase of L then $t \in \mathfrak{Z}$ and if t is a left point of increase then $t \in \overline{\mathfrak{Z}}$. Hence the support of the Borel measure defined by L is included in $\overline{\mathfrak{Z}}$. To show the opposite, we show that if $Q_u = 0$ for some $s < u < t$ then $L(t) > 0$. Look at the first return $\mathfrak{r}_s$ to 0 after s. It is enough to show that $L(\mathfrak{r}_s + \delta) > 0$ for all $\delta > 0$. or that, by the strong Markov property, $L(\delta) > 0$ for all $\delta > 0$. But $L(d_1(x))$ is an exponential random variable with parameter $\Psi(x)$, which is positive, for all small $x > 0$. Hence $L(d_1(x)) > 0$ for all small $x > 0$ a.s. and this completes the proof. $\square$

Theorem 5 (regenerative property of the local time)**.** *The local time L is regenerative on $\mathfrak{Z}$: for any stopping time $\mathfrak{t}$,*

$$L(\mathfrak{t} + \cdot) - L(\mathfrak{t}) \perp\!\!\!\perp \mathscr{F}_{\mathfrak{t}} \text{ a.s. on } \mathfrak{t} \in \mathfrak{Z},$$

and $L(\mathfrak{t}+\cdot) - L(\mathfrak{t}) \stackrel{(d)}{=} L$.

Proof. Let $\mathfrak{t}$ be a stopping time. Assume $\mathfrak{t} < \infty$ and $Q_\mathfrak{t} = 0$ (these are events in $\mathscr{F}_\mathfrak{t}$). Since, by the strong Markov property, the process $Q_{\mathfrak{t}+s}$, $s \geq 0$, is independent of $\mathscr{F}_\mathfrak{t}$ and has the same law as Q it follows that its local time is independent of $\mathscr{F}_\mathfrak{t}$. But, since $Q_\mathfrak{t} = 0$, by the way we constructed the local time for Q, it follows that the local time of $Q_{\mathfrak{t}+s}$, $s \geq 0$, is $L(\mathfrak{t}+s) - L(\mathfrak{t})$, $s \geq 0$. □

Theorem 6 (characterization of L)**.** *Let $A(t)$, $t \geq 0$, be a continuous increasing stochastic process with $A(t)$ being $\mathscr{F}_t$-measurable for all t. Asume that the support of the Borel measure defined by A is $\overline{\mathfrak{Z}}$ and that A has the regenerative property:*

$$\mathbb{P}\big\{\big(Q_{\mathfrak{t}+\cdot},\, A(\mathfrak{t}+\cdot) - A(\mathfrak{t})\big) \in \cdot | \mathscr{F}_\mathfrak{t}\big\} = \mathbb{P}\big\{(Q, A) \in \cdot\big\} \text{ a.s. on } \mathfrak{t} < \infty \text{ and } Q_\mathfrak{t} = 0.$$

Then $A = kL$ for some deterministic $k \geq 0$.

Sketch of proof. The regenerative property can be used to show that $A(d_1(b))$ is memoryless and hence exponential with parameter proportional to $\Psi(b)$, say, $k\Psi(b)$ for some $k \geq 0$. Fixing a and letting $b < a$ be small, we compare $A(d_1(a))$ with $N_b(d_1(a))$ and show that the difference converges to 0 as $b \to 0$. This shows that $A(d_1(a)) = kL(d_1(a))$. Hence $A(t) = kL(t)$ for all $t \in \mathfrak{D}$ and hence $A = kL$. □

Corollary 8. *For any Lévy process X_t, $t \geq 0$, and $Q_t = X_t - \underline{X}_t$, with $L(t)$, $t \geq 0$, the local time at its infimum constructed in Theorem 4, we have that*

$$\int_0^t \mathbf{1}_{Q_s=0}\, ds = \delta\, L(t),$$

for some $\delta \geq 0$.

Proof. Let $A(t)$ denote the left-hand side. Obviously, $A(t)$ is $\mathscr{F}_t$–measurable, it is increasing and continuous in t. The support of the Borel measure defined by A is $\overline{\mathfrak{Z}}$. Let $\mathfrak{t}$ be a stopping time. Then $A(\mathfrak{t}+u) - A(\mathfrak{t}) = \int_\mathfrak{t}^{\mathfrak{t}+u} \mathbf{1}_{Q_s=0}\, ds$ and so the regenerative property of Theorem 6 holds. Therefore A is a multiple of L. □

Note that the multiplicity constant δ might be zero. For example, in the case of Brownian motion W, the set s such that $W_s = \underline{W}_s$ has Lebesgue measure zero.

7.2 Indexing excursions

We found a way to count excursions on intervals $[0, t]$. We indexed excursions according to their durations. We will now find another way to index excursions, one that is compatible with the ordering of $\mathbb{R}$.

We repeat our assumptions: we are dealing with a general Lévy process X_t, $t \geq 0$, with values in $\mathbb{R}$, starting from $X_0 = 0$ and we assume that we are in Case 1, i.e. $\mathbb{P}(\mathfrak{r} = \mathfrak{s} = 0) = 1$.

We have constructed, on the same probability space supporting the Lévy process (that is, as a functional of X) an increasing process $L(t)$, $t \geq 0$, such that it is continuous and such that the Borel measure defined by L has support equal to $\overline{\mathfrak{Z}}$, the set of t such that $X_t = \underline{X}_t$ (equivalently, the points of increase of L are precisely the set $\overline{\mathfrak{Z}}$). We call $L(t)$, $t \geq 0$, the local time of X at its infimum. Intuitively, $L(t)$ is the "amount of time" spent by X at its infimum on the interval $[0,t]$.

Of course, with a mere change of sign, everything we said and will say applies equally well to the points t such that $X_t = \overline{X}_t$. The corresponding "amount of time" spent by X at its supremum is called local time of X at its supremum.

Since L is increasing we can define its generalized inverse

$$L_\nu^{-1} := \inf\{t \geq 0 :\, L(t) > \nu\}, \quad \nu \geq 0.$$

The use of strict inequality $>$ inside the infimum renders L^{-1} right-continuous. Had we used $\geq$ we would have obtained a left-continuous generalized inverse; we would have obtained the left-continuous version of the generalized inverse. Namely,

$$L_{\nu-}^{-1} = \inf\{t \geq 0 :\, L(t) \geq \nu\}, \quad \nu \geq 0.$$

Since we are in Case 1, we have that

$$L_0^{-1} = 0.$$

Notice that $\nu \mapsto L_\nu^{-1}$ is merely right-continuous, not continuous in general; it is continuous only trivial cases. (Which ones?) Notice also that, for all ν and t,

$$L(L^{-1}(\nu)) = \nu, \quad L_{L(t)-}^{-1} \leq t \leq L_{L(t)}^{-1}.$$

The latter is equality only at the points of increase of L.

Lemma 11. *For $\nu \geq 0$, the random times L_ν^{-1} and $L_{\nu-}^{-1}$ are $[0,\infty]$-valued stopping times.*

Proof. Since L is a right-continuous and increasing we have that $L_\nu^{-1} < t \iff \nu < L(t-)$. Since L is continuous, $L(t-) = L(t)$. Hence $\{L_\nu^{-1} < t\} \in \mathscr{F}_t$, so L_ν^{-1} is a stopping time. Since, for all n, $L_{\nu-\frac{1}{n}}^{-1}$ is a stopping time with $L_{\nu-\frac{1}{n}}^{-1} \uparrow L_{\nu-}^{-1}$, the latter is also a stopping time. $\square$

Lemma 12. *For all $t \geq 0$,*

$$\begin{aligned} L_{L(t)-}^{-1} &= \sup\{L_\mu^{-1} :\, L_\mu^{-1} < t\} = \sup\{s < t :\, Q_s = 0\} \\ L_{L(t)}^{-1} &= \inf\{L_\mu^{-1} :\, L_\mu^{-1} > t\} = \inf\{s > t :\, Q_s = 0\} \end{aligned}$$

In particular, $L_{L(t)}^{-1} \in \mathfrak{Z}$ a.s. on $\{L_\nu^{-1} < \infty\}$ and $L_{L(t)-}^{-1} \in \overline{\mathfrak{Z}}$. The component intervals in the decomposition of $\overline{\mathfrak{Z}}$ are precisely the intervals $(L_{\nu-}^{-1}, L_\nu^{-1})$, $\nu > 0$, excluding the empty ones.

Proof. Note that L^{-1} can have at most countably many jumps and so the collection of the nonempty intervals of the form $(L_{\nu-}^{-1}, L_\nu^{-1})$, $\nu > 0$, is countable. The $L_{L(t)}^{-1} = \inf\{L_\mu^{-1} :\, L_\mu^{-1} > t\}$ simply follows from the definition of the right-continuous generalized inverse function. We want to show that this coincides with the time of first return of Q to 0 after time t. Let $D_t := \inf\{s > t :\, Q_s = 0\}$ be this first return time. If $D_t > t$ then, since the support of the measure defined by L is $\overline{\mathfrak{Z}}$, we have that $L(t) = L(D_t-) = L(D_t)$. Hence $D_t \leq L_{L(t)}^{-1}$. If $D_t = \infty$ we are done. Otherwise, D_t is a right point of increase for L, that is, $L(D_t + \varepsilon) > L(D_t) = L(t)$ for all $\varepsilon > 0$. Hence $D_t + \varepsilon \geq L_{L(D_t)}^{-1}$ for all $\varepsilon > 0$ and so $D_t \geq L_{L(D_t)}^{-1}$. We have proved that D_t is $\leq$ and $\geq$ than $L_{L(D_t)}^{-1}$, so they are equal. If $D_t = t$ then t is a right point of increase for L and so $L(t+\varepsilon) > L(t)$ for all $\varepsilon > 0$, so $t \leq L_{L(t)}^{-1}$. Since we always have $t \leq L_{L(t)}^{-1}$, we again have equality. The other identity is proved similarly. $\square$

For function f with at most jump discontinuities, let $\Delta f(t) := f(t+) - f(t-)$.

Corollary 9.

$$L_\nu^{-1} = \sum_{\mu \le \nu} \Delta L_\mu^{-1} + \delta\nu.$$

Proof. The canonical decomposition of the function L_ν^{-1} into a singular part and an absolutely continuous part gives $L_\nu^{-1} = \sum_{\mu\le\nu} \Delta L_\mu^{-1} + C_\nu$, where C_ν is the continuous part. By Lemma 12, $C_\nu = \int_0^{L_\nu^{-1}} \mathbf{1}_{s\in\overline{\mathfrak{Z}}}\, ds = \delta L(L_\nu^{-1}) = \delta\nu$, by Corollary 8. □

In what follows assume that

$$\Psi(\infty) = 0.$$

This ensures that all excursions have finite duration almost surely.

Lemma 13. *The process L_ν^{-1}, $\nu \ge 0$, has stationary and independent increments.*

Proof. This follows from the regenerative property of L. The details are left as an exercise. □

Therefore, L_ν^{-1}, $\nu \ge 0$, is a Lévy process that is, by construction, right-continuous and increasing. Such Lévy processes are called subordinators because they are often used as random time changes. (The composition operation is called subordination, in this business only.)

We can identify the Laplace transform $\mathbb{E}e^{-\lambda L_\nu^{-1}}$ by looking at the Poisson random measure

$$\eta = \sum_{\nu\ge 0:\Delta L_\nu^{-1}>0} \delta_{(\nu,\Delta L_\nu^{-1})}.$$

Note that the sum in the display is countable, so η is a point process. It is a marked point process indexed by $\nu \ge 0$ and having marks in $(0,\infty)$. because it can have at most one point (ν, x) for each ν. It is a Poisson random measure because L^{-1} has independent increments. By Corollary 9,

$$L_\nu^{-1} = \delta\nu + \int_0^\infty x\, \eta([0,\nu], dx) \tag{13}$$

both η and L^{-1} are measurable functions of one another. The mean measure for η is of product form, with Lebesgue being the ν-marginal, i.e.,

$$\mathbb{E}\eta(d\nu, dx) = d\nu\, \Pi(dx),$$

because of stationarity in the ν-coordinate. Let

$$J_a := \inf\{\nu \ge 0 :\, \Delta L_\nu^{-1} > a\}.$$

Then

$$\{J_a > \nu\} = \{\eta([0,\nu]\times(a,\infty)) = 0\}.$$

But $\eta([0,\nu]\times(a,\infty)$ is a Poisson random variable with mean

$$\mathbb{E}\eta([0,\nu]\times(a,\infty) = \nu\Pi(a,\infty)$$

hence

$$\mathbb{P}(J_a > \nu) = e^{-\nu\Pi(a,\infty)}.$$

On the other hand, J_a is the instance of the first jump of L^{-1} of size $> a$. By Lemma 12 this J_a equals $L(t)$ where $t = g_1(a)$ or $t = d_1(a)$. But $L(d_1(a))$ is exponential with rate $\Psi(a)$. Hence

$$\mathbb{P}(J_a > \nu) = \mathbb{P}(L(d_1(a)) > \nu) = e^{-\nu\Psi(a)}.$$

We conclude that

$$\Pi(a, \infty) = \Psi(a).$$

Now recall the famous formula giving the characteristic functional of a Poisson random measure

$$\mathbb{E} \exp \int f d\eta = \exp \int (e^f - 1) d\mathbb{E}\eta,$$

easily obtainable by approximation. Apply this to (13) to obtain

$$\begin{aligned}
\mathbb{E}e^{-\theta L_\nu^{-1}} &= e^{-\nu\delta\theta} \times \exp \iint_{[0,\nu]\times(0,\infty)} (e^{-\theta x} - 1) d\mu \Pi(dx) \\
&= e^{-\nu\delta\theta} \times \exp \nu \int_{(0,\infty)} (e^{-\theta x} - 1) d(-\Psi)(x).
\end{aligned}$$

By integration by parts,

$$\int_{(0,\infty)} (e^{-\theta x} - 1) d(-\Psi)(x) = -\int_{(0,\infty)} (-\theta) e^{-\theta x} (-\Psi(x)) dx = -\theta \int_0^\infty e^{-\theta x} \Psi(x) dx,$$

and so we have

$$\mathbb{E}e^{-\theta L_\nu^{-1}} = \exp\left\{ -\nu\theta\left(\delta + \int_0^\infty e^{-\theta x}\Psi(x)dx\right)\right\}.$$

Remark 4. If $\Psi(\infty) > 0$ then L^{-1} is a subordinator killed at rate $\Psi(\infty)$, indicating that at an exponential variable ν^* with mean $1/\Psi(\infty) < \infty$ the subordinator becomes infinity.

8 Counting and indexing excursions in Case 2

Assume now that Case 2 holds: $\mathfrak{s} > 0$ a.s., and hence $\mathfrak{r} = 0$ a.s. Then, by the strong Markov property, the path of Q consists an "idle period", followed by a "busy period", followed by an "idle period", etc. (Idle means zero. Busy means positive.) That is, there are epochs

$$0 = R_0 < S_1 < R_1 < S_2 < R_2 < \cdots$$

such that $Q > 0$ on each $[S_k, R_k)$ and $Q = 0$ on each $[R_{k-1}, S_k)$. We then have that S_1 is exponentially distributed and $Q_{S_1} > 0$; both these things follow from the strong Markov property; see Exercise 26. Then

$$\mathfrak{Z} = [0, S_1) \cup [R_1, S_2) \cup \cdots$$

Define

$$L(t) := \int_0^t \mathbf{1}_{Q_s=0}\, ds.$$

Note that L is increasing and continuous, that $L(t)$ is measurable with respect to $\mathscr{F}_t$, that the measure defined by L has support $[0, S_1] \cup [R_1, S_2] \cup \cdots = \overline{\mathfrak{Z}}$ and is regenerative in the sense used above:

$$\mathbb{P}\{(Q_{\mathfrak{t}+\cdot}, L(\mathfrak{t}+\cdot) - L(\mathfrak{t})) \in \cdot | \mathscr{F}_\mathfrak{t}\} = \mathbb{P}\{(Q, L) \in \cdot\} \text{ a.s. on } \mathfrak{t} < \infty \text{ and } Q_\mathfrak{t} = 0.$$

We can now study the inverse process L_ν^{-1}, $\nu \geq 0$, and see that $L_0^{-1} > 0$, and that it is a subordinator, possibly killed.

9 Counting and indexing excursions in Case 3

Assume now that Case 3 holds: $\mathfrak{r} > 0$ a.s., and hence $\mathfrak{s} = 0$ a.s. In this case, Q touches 0 at a locally finite set of points $0 = R_0 < R_1 < R_2 < \cdots$ That is, $Q_t = 0$ if and only if $t = R_k$ for some k. This is the case closest to the discrete time one. Indeed, the times R_k are just like the β_k for the discrete-time random walk. The local time process can be nothing else but a version of the counting process of the $(R_k)_{k\geq 0}$. If we do so, we obtain a process with unit jumps. Hence its generalized inverse is also piecewise constant that jumps on the integers. As such it cannot be a Markov process. If we insist that the generalized inverse be a Markov process then we are forced to make replace the unit jumps of the counting process by i.i.d. exponential random variables (see Exercise 26) with some (any) common rate. So we define

$$L(t) = \sum_{k\geq 1} \tau_k \, \mathbf{1}_{R_k \leq t}, \quad t \geq 0,$$

where the τ_k are independent of everything else and i.i.d. exponentials. We lose adaptability, but who cares? We can always increase the filtration. What is important is that L_ν^{-1} is a (killed) subordinator.

10 The Wiener-Hopf factorization for a Lévy process (presto)

Let X_t, $t \geq 0$, be a Lévy process with $\mathbb{E}e^{i\theta X_t} = e^{-t\psi(\theta)}$. Consider an independent exponential random variable T with parameter λ, i.e. $\mathbb{P}(T > t) = e^{-\lambda t}$, $t \geq 0$. We first seek to decompose (T, X_T), just as we did in discrete time (Theorem 1) by considering the path up to the last occurrence of a record before T. As an analytical preliminary, which will be used as a shortcut below, verify that

$$\mathbb{E}e^{-\alpha T + i\theta X_T} = \mathbb{E}e^{(-\alpha - \psi(\theta))T} = \frac{\lambda}{\lambda + \alpha + \psi(\theta)} = \exp\left\{ -\int_0^\infty (1 - e^{-\alpha t - \psi(\theta)t}) t^{-1} e^{-\lambda t} dt \right\}. \quad (14)$$

See Exercise 27. Define next

$$\overline{X}_t := \sup_{s\leq t} X_s, \qquad \underline{X}_t := \inf_{s\leq t} X_s$$

$$G_t = G_t^{up} := \sup\{s < t : X_s = \overline{X}_t\}, \qquad G_t^{down} := \sup\{s < t : X_s = \underline{X}_t\}.$$

We would like to prove something like

$$(T, X_T) \overset{(d)}{=} (G_T^{up}, \overline{X}_T) \dot{+} (G_T^{down}, \underline{X}_T). \quad (15)$$

This might not be quite true in general because records might not occur uniquely. The points t such that $X_t = \overline{X}_t$ are "weak increasing ladder indices". The points t such that $X_t = \underline{X}_t$ are "weak decreasing ladder indices". As we have seen in the discrete-time theory, we cannot have both of them be weak. If one is weak, the dual is strict. Unless, of course, it is the case that records occur uniquely. This happens if and only if X is not a CP (Compound Poisson) process. The following explains this.

Proposition 4. *If X is not a CP process then, for each t, there is a unique instance $G_t^{up} \in [0,t]$ such that $X_{G_t^{up}} = \overline{X}_t$ or $X_{G_t^{up}} = \overline{X}_{t-}$ and a unique instance $G_t^{down} \in [0,t]$ such that $X_{G_t^{down}} = \underline{X}_t$ or $X_{G_t^{down}} = \underline{X}_{t-}$. In other words, the supremum of X on $[0,t]$ is achieved at a unique point. Similarly for the infimum.*

Proof. We show the statement for the supremum. The infimum case will follow by a mere change of sign. We show that for all $s < t$, if there is $s < u < t$ such that $X_u = \overline{X}_u$ then $\overline{X}_s < \overline{X}_t$. Let

$$\mathfrak{r}_+ := \inf\{t > 0 : X_t > 0\} \quad \mathfrak{r}_- := \inf\{t > 0 : X_t < 0\}.$$

Since X is not compound Poisson then $\mathbb{P}(\mathfrak{r}_+ = 0) = 1$ or $\mathbb{P}(\mathfrak{r}_- = 0) = 1$ (or both). Suppose $\mathbb{P}(\mathfrak{r}_+ = 0) = 1$. Then, obviously, $\overline{X}_\varepsilon > 0$ for all $\varepsilon > 0$. Applying the strong Markov property at $\mathfrak{r}_+ \circ \theta_s$ we have that $\overline{X}_t > \overline{X}_{\mathfrak{r}_+ \circ \theta_s} \geq \overline{X}_s$ on the event $\mathfrak{r}_+ \circ \theta_s < t$. This is precisely what we wanted to show. Suppose $\mathbb{P}(\mathfrak{r}_- = 0) = 1$. Then apply the previous argument to $X_t - X_s$, $s \geq 0$. $\square$

We now show something more general than (15), something that holds regardless of whether X is PC or not.

Theorem 7 (path decomposition in continuous time)**.** *Let X_t, $t \geq 0$, be any Lévy process in $\mathbb{R}$ and let T be an independent exponential random variable with rate λ. Then*

$$(T,\, X_T) \overset{\text{(d)}}{=} (G_T,\, \overline{X}_T) \dot{+} (T - G_T,\, X_T - \overline{X}_T), \tag{16}$$

Proof. We shall consider two cases by considering the time

$$\mathfrak{r} := \inf\{t > 0 :\, \overline{X}_t - X_t = 0\}.$$

By Blumenthal's 0-1 law, either $\mathfrak{r} > 0$ a.s., or $\mathfrak{r} = 0$ a.s.
<u>Case 1: $\mathfrak{r} > 0$ a.s.</u> Then the times t such that $\overline{X}_t - X_t = 0$ form a locally finite set and so we can enumerate them:

$$0 = \mathfrak{r}_0 < \mathfrak{r} = \mathfrak{r}_1 < \mathfrak{r}_2 < \cdots$$

They are all stopping times. In fact, they form iterates of $\mathfrak{r}$ as in the discrete-time case. We shall prove that

$$\mathcal{A} := (X_t,\, 0 \leq t \leq G_T) \perp\!\!\!\perp (X_t - X_{G_T},\, G_T \leq t < T) =: \mathcal{B}.$$

From this, our claim will follow because $(G_T, \overline{X}_T) = (G_T, X_{G_T})$ is a function of the random element on the left, while $(T - G_T,\, X_T - \overline{X}_T)$ is a function of the random element on the right. To prove the claim in the display, we shall prove that

$$\mathbb{E} f(\mathcal{A}) g(\mathcal{B}) = \mathbb{E} f(\mathcal{A})\, \mathbb{E} g(\mathcal{B})$$

for bounded measurable functions f and g. This is a consequence of the strong Markov property and the exponentiality:

$$\begin{aligned}
\mathbb{E}f(\mathcal{A})g(\mathcal{B}) &= \sum_{n\geq 0}\mathbb{E}[f(\mathcal{A})g(\mathcal{B}); \mathfrak{r}_n < T \leq \mathfrak{r}_{n+1}] \\
&= \sum_{n\geq 0}\mathbb{E}[f(X_t,\, 0\leq t\leq \mathfrak{r}_n)g(X_t - X_{\mathfrak{r}_n},\, \mathfrak{r}_n \leq t < T); \mathfrak{r}_n < T \leq \mathfrak{r}_{n+1}] \\
&= \sum_{n\geq 0}\mathbb{E}\int_{\mathfrak{r}_n}^{\mathfrak{r}_{n+1}} f(X_t,\, 0\leq t\leq \mathfrak{r}_n)g(X_t - X_{\mathfrak{r}_n},\, \mathfrak{r}_n \leq t < s)\lambda e^{-\lambda s}ds \\
&= \sum_{n\geq 0}\mathbb{E}[f(X_t,\, 0\leq t\leq \mathfrak{r}_n)e^{-\lambda \mathfrak{r}_n}] \times \mathbb{E}\int_0^{\mathfrak{r}} g(X_t,\, 0\leq t < s)\lambda e^{-\lambda s}ds \\
&= \mathbb{E}f(\mathcal{A})\,\mathbb{E}g(\mathcal{B}).
\end{aligned}$$

Case 2: $\mathfrak{r} = 0$ a.s. We know that there is a continuous local time process $L(t)$, $t \geq 0$, for the strong Markov process $\overline{X}_t - X_t$, $t\geq 0$, "measuring" the amount of time that the latter spends at 0 (or that X spends at its supremum). Let m be a large positive integer and let

$$N_m = [mL(T)] \quad \text{(integer part)}.$$

Then

$$\frac{N_m}{m} < L(T) < \frac{N_m+1}{m} \text{ a.s.}$$

Hence

$$L^{-1}_{N_m/m} \leq T \leq L^{-1}_{(N_m+1)/m} \text{ a.s.}$$

Using the regenerative property of L and the argument from the first case obtain that

$$(X_t,\, 0\leq t\leq L^{-1}_{N_m/m}) \perp\!\!\!\perp (X_t - X_{L^{-1}_{N_m/m}},\, L^{-1}_{N_m/m} \leq t < T).$$

As $m\to\infty$, we have that $N_m/m \uparrow L(T)$ and $L^{-1}_{N_m/m} \uparrow L^{-1}_{L(T)-} = G_T$. Hence

$$(X_t,\, 0\leq t\leq L^{-1}_{N_m/m}) \to (X_t,\, 0\leq t < G_T)$$

$$(X_t - X_{L^{-1}_{N_m/m}},\, L^{-1}_{N_m/m} \leq t < T) \to (X_t - X_{G_T-},\, G_T \leq t < T).$$

We thus have

$$\mathcal{A} := (X_t,\, 0\leq t < G_T) \perp\!\!\!\perp (X_t - X_{G_T-},\, G_T\leq t < T) =: \mathcal{B}.$$

From this we get

$$(G_T,\, X_{G_T-}) \perp\!\!\!\perp (T - G_T,\, X_T - X_{G_T-}),$$

But our claim is that $(G_T, \overline{X}_T) \perp\!\!\!\perp (T - G_T, X_T - \overline{X}_T)$. This will follow once we show that

$$X_{G_T-} = \overline{X}_T \text{ a.s.}$$

There are two cases to consider: either X is continuous at G_T or not. If X is continuous at G_T then $X_{G_T-} = X_{G_T} = \overline{X}_{G_T} = \overline{X}_T$. If X is not continuous at G_T then $X_{G_T-} > X_{G_T}$ or $X_{G_T-} < X_{G_T}$. If $X_{G_T-} > X_{G_T}$ then $X_{G_T-} = \overline{X}_{G_T-} = \overline{X}_{G_T} = \overline{X}_T$. If $X_{G_T-} < X_{G_T}$ we reach a contradiction by considering the locally finite set of (stopping) times t at which $X_t - X_{t-} > \varepsilon > 0$ and by applying the strong Markov property. $\square$

Theorem 8. *Let X_t, $t \geq 0$, be a general Lévy process If T is an independent exponential random variable with rate λ, we can identify the laws of the stochastic decomposition (16) as follows:*

$$\mathbb{E}e^{-\alpha G_T - \beta \overline{X}_T} = \exp\left\{ -\int_{[0,\infty)} \int_{\mathbb{R}} (1 - e^{-\alpha t - \beta x}) t^{-1} e^{-\lambda t} \mathbb{P}(X_t \in dx) dt \right\}$$

$$\mathbb{E}e^{-\alpha(T - G_T) + \beta(X_T - \overline{X}_T)} = \exp\left\{ -\int_{(-\infty,0)} \int_{\mathbb{R}} (1 - e^{-\alpha t + \beta x}) t^{-1} e^{-\lambda t} \mathbb{P}(X_t \in dx) dt \right\}, \quad \alpha, \beta \geq 0.$$

The theorem becomes more transparent once we write and prove it in the case where X is not a CP process. We shall do this.

Proof of the theorem in the non-CP case. If X is not a CP process then we have several things happening at the same time. First of all, as proved in Proposition 4, there are no ties between records. Using this and duality, we can show that

$$(t - G_t, X_t - \overline{X}_t) \overset{\text{(d)}}{=} (G_t^{down}, \underline{X}_t).$$

Therefore, (15) holds. We repeat this here:

$$(T,\, X_T) \overset{\text{(d)}}{=} (G_T^{up},\, \overline{X}_T) \dotplus (G_T^{down},\, \underline{X}_T).$$

Moreover, $\mathbb{P}(X_t = 0) = 0$. Note that the law of (T, X_T) has been identified at (14). We rewrite this as

$$\mathbb{E}e^{-\alpha T + i\theta X_T} = \exp\left\{ \int_0^\infty (e^{-\alpha t - \psi(\theta)t} - 1) t^{-1} e^{-\lambda t} dt \right\} = \exp\left\{ \mathbb{E} \int_0^\infty (e^{-\alpha t + i\theta X_t} - 1) t^{-1} e^{-\lambda t} dt \right\}$$

$$= \exp\left\{ \mathbb{E}\left[\mathbf{1}_{X_t > 0} \int_0^\infty (e^{-\alpha t + i\theta X_t} - 1) t^{-1} e^{-\lambda t} dt \right] \right\} \exp\left\{ \mathbb{E}\left[\mathbf{1}_{X_t < 0} \int_0^\infty (e^{-\alpha t + i\theta X_t} - 1) t^{-1} e^{-\lambda t} dt \right] \right\},$$

So the first term in the product equals $\mathbb{E}e^{-\alpha G_T^{up} + i\theta \overline{X}_T}$, while the second term equals $\mathbb{E}e^{-\alpha G_T^{down} + i\theta \underline{X}_T}$. □

Corollary 10. *For a general Lévy process X_t, $t \geq 0$, and T an independent exponential$[\lambda]$ variable, the law of $G_t = \sup\{s < t : X_t = \overline{X}_t\}$ is characterized by*

$$\mathbb{E}e^{-\alpha G_T} = \exp\left\{ \int_0^\infty (e^{-\alpha t} - 1) t^{-1} e^{-\lambda t} \mathbb{P}(X_t \geq 0) dt. \right\}$$

11 Excursion theory (prestissimo)

Let Q_t, $t \geq 0$, be (as earlier) a strong Markov processwith respect to a right-continuous filtration $\mathscr{F}_t$, $t \geq 0$. We assume that the process starts from 0, returns to it immediately and also leaves it immediately We constructed a continuous local time process $L(t)$, $t \geq 0$, supported on the set $\overline{\mathfrak{Z}}$, the closure of the set $\{t : Q_t = 0\}$ such that it regenerates, along with Q, on this set. The right-continuous inverse function L_ν^{-1}, $\nu \geq 0$, is such that $L_{\nu-}^{-1}$ and L_ν^{-1}

are stopping times and, for any real time t, the excurstion straddling t starts at $L^{-1}_{L(t)-}$ and ends at $L^{-1}_{L(t)}$ if the latter is finite.

We now let EXC be the set where excursions take values in. This is a nice Polish space. It contains càdlàg functions of either finite or infinite duration. We break the space into pieces, letting $\mathrm{EXC}(u)$ be the set of excursions of duration at least u:

$$\mathrm{EXC} = \bigcup_{u>0} \mathrm{EXC}(u).$$

Note that $b < a \Rightarrow \mathrm{EXC}(b) \supset \mathrm{EXC}(a)$, thus $\mathrm{EXC}(u)$ increases as u decreases. We shall put a natural measure ϖ on EXC by putting a natural measure ϖ_u on $\mathrm{EXC}(u)$ for each u and by showing that the family is compatible.

Consider now the excursions of the process Q_t, $t \geq 0$, that are in $\mathrm{EXC}(u)$. We saw that they are countably many and are ordered:

$$\mathcal{E}_1(u),\, \mathcal{E}_2(u)\,, \ldots,$$

where $\mathcal{E}_k(u)$ is the excursion on $(g_k(u), d_k(u))$. We define

$$\varpi_u(A) := \Psi(u)\, \mathbb{P}(\mathcal{E}_1(u) \in A), \quad A \subset \mathrm{EXC}(u) \text{ (measurable)}.$$

Therefore,

$$\varpi_u(\mathrm{EXC}(u)) = \Psi(u).$$

To see that the family ϖ_u, $u > 0$, defines a unique measure ϖ on EXC, we must show that, for $b < a$ (whence $\mathrm{EXC}(b) \supset \mathrm{EXC}(a)$) the restriction of ϖ_b from $\mathrm{EXC}(b)$ to $\mathrm{EXC}(a)$ is ϖ_a. This is equivalent to

$$\varpi_b(A) = \varpi_a(A) \text{ for } A \subset \mathrm{EXC}(a).$$

By the second part of Lemma 9 we have that $N_b(g_1(a))$ is independent of the process Q after $g_1(a)$. Hence

$$\mathbb{P}(\mathcal{E}_1(a) \in A,\, N_b(g_1(a)) = 0) = \mathbb{P}(\mathcal{E}_1(a) \in A)\, \mathbb{P}(N_b(g_1(a)) = 0) = \mathbb{P}(\mathcal{E}_1(a) \in A)\, \frac{\Psi(a)}{\Psi(b)}.$$

But $N_b(g_1(a)) = 0$ means that $\mathcal{E}_1(a) = \mathcal{E}_1(b)$. Hence

$$\mathbb{P}(\mathcal{E}_1(a) \in A,\, N_b(g_1(a)) = 0) = \mathbb{P}(\mathcal{E}_1(b) \in A,\, N_b(g_1(a)) = 0).$$

Also, $N_b(g_1(a))$ means that the duration of $\mathcal{E}_1(b)$ is at least a. Hence, for $A \subset \mathrm{EXC}(a)$, we immediately get that

$$\mathbb{P}(\mathcal{E}_1(b) \in A,\, N_b(g_1(a)) = 0) = \mathbb{P}(\mathcal{E}_1(b) \in A).$$

Hence, for $A \subset \mathrm{EXC}(a)$,

$$\mathbb{P}(\mathcal{E}_1(b) \in A) = \mathbb{P}(\mathcal{E}_1(a) \in A)\, \frac{\Psi(a)}{\Psi(b)}$$

and this is precisely the compatibility condition.

We now consider, for each $\nu > 0$, such that $\Delta L_\nu^{-1} > 0$, the excursion of the process on the interval $(L_{\nu-}^{-1}, L_\nu^{-1})$:

$$\mathcal{E}_\nu := (Q_t,\ L_{\nu-}^{-1} < t < L_\nu^{-1}).$$

Recall that, thanks to the indpendent increments of L^{-1},

$$\eta = \sum_{\nu \geq 0: \Delta L_\nu^{-1} > 0} \delta_{(\nu, \Delta L_\nu^{-1})}.$$

is a Poisson random measure on $[0, \infty) \times (0, \infty)$ with mean measure $\mathbb{E}\eta(d\nu, dx) = d\nu\, \Pi(dx)$, where Π is the measure defined by $\Pi(B) = -\int_B \Psi(dx)$, $B \subset (0, \infty)$. We shall consider instead the random measure

$$\xi = \sum_{\nu \geq 0: \Delta L_\nu^{-1} > 0} \delta_{(\nu, \mathcal{E}_\nu)}$$

on the space $[0, \infty) \times \mathrm{EXC}$. The first thing to observe is that $\xi \mapsto \eta$ is a deterministic map. Thus η is a projection of ξ onto a smaller space. The second thing to observe is that ξ is also a Poisson random measure, thanks to the regenerativity of L. The third thing to observe is that the mean measure of ξ is given by

$$\mathbb{E}\xi(d\nu, d\varepsilon) = d\nu\, \varpi(d\varepsilon).$$

We thus have

Theorem 9 (Itô's theorem)**.** *The measure $\xi = \sum_{\nu \geq 0: \Delta L_\nu^{-1} > 0} \delta_{(\nu, \mathcal{E}_\nu)}$ is a (possibly killed) Poisson random measure with mean measure $\mathbb{E}\xi(d\nu, d\varepsilon) = d\nu\, \varpi(d\varepsilon)$.*

The term "possibly killed" refers to the fact that $L(\infty)$ may not be equal to ∞.

The so-called compensation formula can be obtained easily. Let $F_t = (F_t(\varepsilon), \varepsilon \in \mathrm{EXC})$ be a left-continuous adapted stochastic process with values in the space of bounded and measurable functions on EXC. Let G be the set of left endpoints of excursion intervals (the set $\{\inf I : I \in \mathscr{I}\}$). For $g \in G$, let ε_g be the excursion starting at g. In particular, $\varepsilon_{L_\nu^{-1}} = \mathcal{E}_\nu$. We then have

$$\mathbb{E} \sum_{g \in G} F_g(\varepsilon_g) = \mathbb{E} \int_0^\infty dL(s) \int_{\mathrm{EXC}} F_s(\varepsilon) \varpi(\varepsilon).$$

Indeed, first observe that $\sum_{g \in G} F_g(\varepsilon_g) = \sum_{\nu \geq 0: \Delta L_\nu^{-1} > 0} F_{L_\nu^{-1}}(\mathcal{E}_\nu)$, then use the compensation formula for marked Poisson processes:

$$\mathbb{E} \sum_{g \in G} F_g(\varepsilon_g) = \mathbb{E} \sum_{\nu \geq 0: \Delta L_\nu^{-1} > 0} F_{L_\nu^{-1}}(\mathcal{E}_\nu) = \mathbb{E} \int_0^\infty d\nu \int_{\mathrm{EXC}} F_{L_\nu^{-1}}(\varepsilon) \varpi(d\varepsilon)$$

and finally change variables in the last Lebesgue-Stieltjes integral.

12 CHEAT SHEET (a piacere)

1. If Φ is a Poisson random measure on a general measurable space with σ-finite intensity measure $\mathbb{E}\Phi(\cdot)$ then the law of Φ is characterized by the map $f \mapsto \mathbb{E} \exp \int f d\Phi$ with f in a nice class of functionals (e.g. bounded measurable), and it is a consequence of the independent increments property ($A \cap B = \varnothing \Rightarrow \Phi(A) \perp\!\!\!\perp \Phi(B)$) that the following explicit formula holds:

$$\mathbb{E} \exp \int f d\Phi = \exp \int (e^f - 1) d\mathbb{E}\Phi.$$

If Φ lives on a "nice" space (which is at least Borel, that is, a space which such that there is an invertible function from the space onto a Borel subset of $\mathbb{R}$ with the property that both the function and its inverse are measurable; for example, $\mathbb{R}^d$ is a Borel space; the space $D(I)$ of càdlàg functions on an interval I is a Borel space; more generally, any Polish space is a Borel space), then Φ can be considered as a random discrete locally finite set. For example, if Φ a Poisson random measure on a Euclidean space then requiring that Φ is a random measure with values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ with the independent increments property we immediately obtain that $\Phi(A)$ is a Poisson random variable with mean $0 \leq \mathbb{E}\Phi(A) \leq \infty$.

2. First two moments:

$$\mathbb{E} \int f d\Phi = \int f d\mathbb{E}\Phi, \quad \operatorname{var} \int f d\Phi = \int f^2 d\mathbb{E}\Phi.$$

3. A random measure Φ on a product space $I \times M$, where I is a Borel space, is said to be marked with marks in M and index in I if $\Phi(\{t\} \times M) = 0$ a.s. for all $t \in I$. This actually tells us that if $(t, m), (t, m')$ are points of Φ then $m = m'$: there can be no two points with the same I-coordinate. A very important theorem due to Erlang and Lévy says that such a Φ is a Poisson random measure if and only if it has independent increments. In this case, the measure $A \mapsto (\mathbb{E}\Phi)(A, B)$ has no atoms. If $t \in I$ is such that (t, m) is an atom of Φ, let $m(t)$ be equal to this m. Otherwise, set m to a graveyard. The collection of random variables (they are random variables!) $m(t)$, $t \in I$, is referred to as "Poisson point process" by some people–an unfortunate terminology.

4. If X_t, $t \geq 0$, is a collection of random variables in $\mathbb{R}^d$ with independent increments that are continuous in a weak sense (in probability) and have no fixed jumps, then martingale theory tells us that there is a version of them with càdlàg paths. Using the notation $\Delta X_t := X_t - X_{t-}$, define

$$\Phi = \sum_{t : \Delta X_t \neq 0} \delta_{(t, \Delta X_t)}.$$

Note that the sum is a countable sum and that Φ a marked random measure on $[0, \infty) \times \mathbb{R}$ with marks in $\mathbb{R}$. The Erlang-Lévy theorem tells us that Φ is a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$.

5. First property of $E\Phi$, that is true by Poissonicity:

$$\mathbb{E}\Phi([0, t] \times \{x : |x| \geq 1\}) < \infty.$$

6. An independent increments process is simple if $\sum_{s \leq t} \mathbf{1}(\Delta X_s \neq 0) < \infty$ a.s., for all t. A simple independent increments process without continuous component is called compound Poisson.

7. Using the fact that two martingales that cannot jump together are independent, we can show that the "big" jumps process $X_t^\varepsilon := \sum_{s \leq t} \Delta X_s \mathbf{1}(|\Delta X_s| > \varepsilon)$ is independent of the "small" jumps process $X_t - X_t^\varepsilon$.

8. Second property of $\mathbb{E}\Phi$:

$$\iint_{s \leq t, |x| \leq 1} |x|^2 \mathbb{E}\Phi(ds, dx) < \infty.$$

This follows from an estimate of the characteristic function of X_t^ε when $\varepsilon \downarrow 0$:

$$\left|\mathbb{E}e^{i\theta X_t^\varepsilon}\right| = \left|\exp \iint_{\substack{s\le t\\|x|>\varepsilon}} (e^{i\theta x}-1)\mathbb{E}\Phi(ds,dx)\right| = \exp - \iint_{\substack{s\le t\\|x|>\varepsilon}} (1-\cos(\theta x))\mathbb{E}\Phi(ds,dx).$$

If $\iint_{s\le t,|x|\le 1}|x|^2\mathbb{E}\Phi(ds,dx) = \infty$ then, owing to $1-\cos(\theta x) = O(|x|^2)$, we have $Ee^{i\theta X_t^\varepsilon} \to 0$ as $\varepsilon \downarrow 0$, a contradiction since $|\mathbb{E}e^{i\theta X_t^\varepsilon}| \ge |\mathbb{E}e^{i\theta X_t}|$.

9. Whereas sums of big jumps of magnitude, say, larger than 1,

$$X_t^1 = \sum_{s\le t}\Delta X_s \mathbf{1}(|\Delta X_s|>1) = \iint_{s\le t,|x|>1} x\Phi(ds,dx),$$

can be defined without any problem, this is not the case with small jumps. However,

$$Y_t^1 = \iint_{s\le t,|x|\le 1} x[\Phi(ds,dx)-\mathbb{E}\Phi(ds,dx)]$$

does make sense, owing to the second property of $\mathbb{E}\Phi$. To abbreviate, write $X^1 = \int_{|x|>1}xd\Phi$, $Y^1 = \int_{|x|\le 1}x[d\Phi - d\mathbb{E}\Phi]$. Since we <u>cannot</u> write $X^1+Y^1 = \int_{|x|>1}xd\Phi + \int_{|x|\le 1}xd\Phi - \int_{|x|\le 1}xd\mathbb{E}\Phi = \int_{\mathbb{R}_0^d}xd\Phi - \int_{|x|\le 1}xd\mathbb{E}\Phi$, from which it would have been obvious that X and X^1+Y^1 have the same jumps, we <u>must</u> make sure about the validity of the latter in a different way. To this end, note that

$$Y_t^{\varepsilon,1} = \int_{s\le t,\varepsilon<|x|\le 1} x[\Phi(ds,dx)-\mathbb{E}\Phi(ds,dx)]$$

is a simple II process that converges to Y_t^1 in the sense that

$$E\sup_{s\le t}|Y_s^{\varepsilon,1}-Y_s^1|^2 \xrightarrow[\varepsilon\to 0]{} 0.$$

The reason is:

$$E\sup_{s\le t}|Y_s^{\varepsilon,1}-Y_s^1|^2 \le_{(a)} 2E|Y_t^{\varepsilon,1}-Y_t^1|^2 =_{(b)} 2\iint_{s\le t,|x|\le\varepsilon}|x|^2E\eta(ds,dx).$$

(a) is Doob's inequality; (b) is the variance formula for Poisson random measures. Since X^1+Y^1 has the same jumps as X, the process $Z := X-(X^1+Y^1)$ is a.s. continuous, with continuous mean and covariance and independent increments. Hence Z is Gaussian. (This requires a version of the central limit theorem.) Since it has no jumps, it is independent of $X^1+Y^{\varepsilon,1}$, for all $\varepsilon>0$. Hence Z and X^1+Y^1 are independent.

10. Putting things together, we have found that: There exists a Poisson random measure Φ on $[0,\infty)\times\mathbb{R}^d$ such that

$$\iint_{[0,t]\times\mathbb{R}^d}|x|^2\wedge 1\ \mathbb{E}\Phi(ds,dx)<\infty,$$

and a Gaussian process Z with independent increments and continuous mean and covariance, such that

$$X_t = Z_t + \iint_{s\le t,|x|\le 1}x[\Phi(ds,dx)-\mathbb{E}\Phi(ds,dx)] + \iint_{s\le t,|x|>1}x\Phi(ds,dx).$$

11. Specializing to Lévy processes, we find (due to translation invariance) that in addition we need $\mathbb{E}Z_t = bt$, for some $b\in\mathbb{R}^d$, $\mathbb{E}Z_s\otimes Z_t = \sigma(s\wedge t)$, for some $d\times d$ positive semidefinite matrix σ, and $\mathbb{E}\Phi(ds,dx) = ds\Pi(dx)$, for some σ-finite measure Π on $\mathbb{R}^d$ with no mass at 0 such that

$$\int_{\mathbb{R}^d}|x|^2\wedge 1\ \Pi(dx)<\infty.$$

12. From the above we read immediately the characteristic function:

$$\mathbb{E}e^{i\theta X_t} = e^{-t\psi(\theta)} = \exp\left\{i\langle\theta,b\rangle - \frac{1}{2}\langle\theta,\sigma\theta\rangle + \int_{\mathbb{R}^d}[e^{i\langle\theta,x\rangle}-1-i\langle\theta,x\rangle\mathbf{1}(|x|\le 1)]\ \Pi(dx)\right\}.$$

13. Let μ be an infinitely divisible (ID) probability measure on $\mathbb{R}^d$: for all n, $\mu = \mu_n^{\star n}$ and μ_n is also probability. It can be shown that $\mu_n \Rightarrow \delta_0$. Then $\widehat{\mu}_n(\theta) = \int e^{i\theta x}\mu_n(dx) \xrightarrow[n\to\infty]{} 1$, uniformly in $-\varepsilon \le \theta \le \varepsilon$, implying that $\widehat{\mu}(\theta) \not\equiv 0$. Hence $\psi(\theta) := -\log \widehat{\mu}(\theta)$ is well defined function from $\mathbb{R}$ to $\mathbb{C}$ and we can choose a continuous branch of it. So $\widehat{\mu}(\theta) = e^{-\psi(\theta)}$. Similarly, $\widehat{\mu}_n(\theta) = e^{-\psi_n(\theta)}$ and $\psi(\theta) = \frac{1}{n}\psi_n(\theta)$. Since $\widehat{\mu}_n(\theta) = e^{-\frac{1}{n}\psi(\theta)}$ is a ch.f., so is any integer power $\widehat{\mu}_n(\theta)^m = e^{-\frac{m}{n}\psi(\theta)}$. By Bôchner's theorem, $e^{-t\psi(\theta)}$ is a ch.f. for each t. Let μ_t be the probability corresponding to it. Since $e^{-t\psi(\theta)} = e^{-(t-t_k)\psi(\theta)} \cdots e^{-(t_2-t_1)\psi(\theta)}e^{-t_1\psi(\theta)}$, we have, by Kolmogorov's consistency, the existence of a collection $(X_t, t \ge 0)$ of random variables with independent increments. Also, $e^{-t\psi(\theta)} \xrightarrow[t\to 0]{} 1$, the process is Lévy. Hence the Lévy-Khinchine formula holds. We conclude that the Lévy-Khnichine formula holds for any ID distribution.

14. The paths of the Lévy process have bounded variation $\iff$ $\sigma = 0$ and $\int |x| \wedge 1 \; \Pi(dx) < \infty$. In this case, there is no need for compensation for the small jumps, so we can write:

$$X_t = bt + \int_0^t \int_{\mathbb{R}^d} x \; \eta(ds, dx)$$
$$Ee^{i\theta X_1} = \exp\left[i\langle\theta, b\rangle + \int_{\mathbb{R}^d} [e^{i\langle\theta,x\rangle} - 1] \; \Pi(dx)\right]$$

The constant b is referred to as the drift. When $d = 1$, we can decompose X as the difference of two mutually singular increasing processes:

$$X_t = bt + \int_0^t \int_{\mathbb{R}_+} x \; \Phi(ds, dx) - \int_0^t \int_{\mathbb{R}_-} (-x) \; \Phi(ds, dx)$$

(Put b together with the first or the second integral, according to its sign.) Each of the last two terms is called subordinator.

15. A subordinator should be thought of as continuous-time analog of the points of a renewal process. Thus the inverse of a subordinator corresponds to the counting process of the renewal process. If X is a subordinator then

$$X_t = \delta t + \int_0^t \int_{\mathbb{R}_+} x\Phi(ds, dx).$$

The Laplace transform $\mathbb{E}e^{-\alpha X_t}$ exists for at least all $\alpha \ge 0$ and

$$\mathbb{E}e^{-\alpha X_t} = e^{-t\psi(i\alpha)} =: e^{-t\varphi(\alpha)},$$

where

$$\psi(\theta) = -i\delta\theta + \int_{\mathbb{R}_+} (1 - e^{i\theta x})\Pi(dx)$$

so,

$$\varphi(\alpha) = \psi(i\alpha) = \delta\alpha + \int (1 - e^{-\alpha x})\Pi(dx).$$

Just as a renewal process can be defective if the probability of the interrenewal time being equal to infinity is positive, so can a Lévy process be defective, or "killed" as is commonly said. "Killing a subordinator at rate λ" means "consider the subordinator up to an independent exponential random variable τ_λ of rate λ". Note that we do not necessarily wish to kill a subordinator for fun but that the killing may happen out of necessity rather than caprice. (See the analogy with the découpage de Lévy and its construction using an independent geometric random variable.) Killing at rate λ simply means add λ to the Laplace exponent:

$$\mathbb{E}[e^{-\alpha X_t}; t < \tau_\lambda] = \exp\left\{-\left(\lambda + \delta\alpha + \int (1 - e^{-\alpha x})\Pi(dx)\right)\right\}.$$

16. A compound Poisson process is a random walk $S_n = \xi_1 + \cdots + \xi_n$ where the time variable n is substituted by a Poisson process N_t (of rate, say, λ):

$$X_t = S_{N_t}.$$

If F is the law of ξ_1, we have

$$\mathbb{E}e^{i\theta X_t} = \exp\left\{t \int (e^{i\theta x} - 1)\lambda F(dx)\right\}.$$

So $\Pi(dx) = \lambda F(dx)$ is a finite measure with total mass $\Pi(\mathbb{R}) = \lambda$.

17. The term "drift" for a general Lévy process means nothing. It is defined if and only if the process has paths of bounded variation. There is a discrepancy from random walk terminology here. For, in discrete time, the drift of the random walk $S_n = \xi_1 + \cdots + \xi_n$ often refers to $\mathbb{E}S_n = (\mathbb{E}\xi_1)n$, if it exists. A Lévy process is integrable, i.e. $\mathbb{E}X_t < \infty$ if and only if $\int_{|x|>1} |x| \, \Pi(dx) < \infty$. For an integrable Lévy process, it is often convenient to separate the mean. That is, write it as

$$X_t = \mu t + \sigma B_t + \int_0^t \int x \, [\eta(ds, dx) - ds\Pi(dx)]$$

where μ is the mean per unit of time, σB_t the Brownian component, and the rest is the zero-mean jump part. Then

$$Ee^{i\theta X_1} = \exp\left\{ i\langle\theta, \mu\rangle - \frac{1}{2}\langle\theta, \sigma\theta\rangle + \int_{\mathbb{R}^d} [e^{i\langle\theta, x\rangle} - 1 - i\langle\theta, x\rangle] \, \Pi(dx) \right\}$$

When (i) the jump part has finite variation ($\int |x| \wedge 1 \; \Pi(dx) < \infty$) and (ii) the first moment is finite ($\int_{|x|>1} |x| \; \Pi(dx) < \infty$), we are in the happy situation that

$$\int |x|\Pi(dx) < \infty$$

and then we can either decide to exhibit the mean (as above) or exhibit the drift:

$$X_t = \left[\mu - \int_{\mathbb{R}^d} x \; \Pi(dx)\right] t + \sigma B_t + \int_0^t \int_{\mathbb{R}^d} x \; \eta(ds, dx)$$

18. Duality: In dicrete time, duality means taking the sum $\xi_1 + \cdots + \xi_n$ in reverse order: $\xi_1 + \cdots + \xi_n$. Then X_t, $0 \le t \le n$, has the same law as $X_n - X_t$, $0 \le t \le n$, for each n. If we think of increments $X_t - X_s$ rather than values, duality becomes more transparent. For each t, the collection X_s, $0 \le s \le t$, of random variables has the same finite-dimensional distributions as the collection $X_t - X_{t-s}$, $0 \le s \le t$. As processes with values in $D[0, t]$, the two collections do not have identical law (simply because the second one is left-continuous), so the remedy is to replace the second one by $X_t - X_{(t-s)-}$, $0 \le s \le t$. This does not destroy the finite-dimensional distributions but makes it right-continuous. Then we have that the two processes are identical in law as random elements of $D[0, t]$ for each $t \ge 0$. Duality implies that $(\overline{X}_t, \overline{X}_t - X_t) \overset{\text{(d)}}{=} (X_t - \underline{X}_t, -\underline{X}_t)$. This leads to the following: $\mathbb{P}(\overline{X}_t = X_t) = 0$ for all t if and only if X enters $(-\infty, 0)$ immediately. Indeed, $\mathbb{P}(\overline{X}_t = X_t) = 0$ for all $t \iff \mathbb{P}(-\underline{X}_t = 0) = 0$ for all $t \iff \mathbb{P}(\underline{X}_t < 0) = 1$ for all t.

19. Let Φ be a marked Poisson process on $[0, \infty) \times M$ with mean measure $\mathbb{E}\Phi(dt, dm) = dt \, \pi(dm)$. If (t, m) is an atom of Φ then m is uniquely defined and, as above, denoted by $m(t)$. Introduce the natural σ-algebra $\mathscr{G}_t$ generated by the restriction on $[0, t]$. Let $F_t = (F_t(m), m \in M)$, $t \ge 0$, be a left-continuous stochastic process with values in the space of bounded and measurable functions on M, such that F_t is measurable with respect to $\mathscr{G}_t$ for each t. Then

$$\mathbb{E} \iint F_t(m)\Phi(dt, dm) = \mathbb{E} \sum_{\substack{t \ge 0 \\ \Phi\{(t,m)\}=1}} F_t(m(t)) = \mathbb{E} \int_0^\infty dt \int_M F_t(m)\pi(dm)$$

If π is a finite measure then we can enumerate the points of Φ in increasing order of the t-coordinate: $(T_1, Z_1), (T_2, Z_2), \ldots$, with $T_1 < T_2 < \cdots$ Suppose also that $F : [0, \infty) \times M \to \mathbb{R}_+$ is measurable. We then have

$$\mathbb{E} \sum_n F(T_n, Z_n) = \int_0^\infty dt \int_M F(t, m)\pi(dm).$$

Such a formula is provable from within the framework of Palm probabilities as well. It is also known as Campbell's formula.

20. Let Φ be a random measure on some Polish space E. This means that Φ is a random variable on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in the space of measures on E. Assume that the mean measure $\Lambda(B) := \mathbb{E}\Phi(B)$ is locally finite. We can then capture the notion of conditioning on the measure Φ having mass at a point $x \in E$ by using a Radon-Nikodým derivative. Observe that for all $A \in \mathscr{F}$ the so-called Campbell measure $B \mapsto \mathbb{E}\Phi(B)\mathbf{1}_A$ is absolutely continuous with respect to Λ. Then the Radon-Nikodým derivative of the Campbell measure with respect to Λ exists and is denoted by $\mathbb{P}^x(A)$. Thus, formally, $\mathbb{P}^x(A) = \mathbb{E}[\Phi(dx)\mathbf{1}_A]/\Lambda(dx)$. Then $\mathbb{P}^x$ is a probability measure on Ω and is called the Palm probability of $\mathbb{P}$ with respect to Φ at the point x. If E is a group and if the law of Φ is invariant with respect to the group actions then $\mathbb{P}^x$ is uniquely determined by P^0 where 0 is taken to be the neutral element of the group. Cambell's formula follows by writing the Radon-Nikodým derivative in integral form.

13 EXERCISES

1. Think about the statement $X_1 \overset{(d)}{=} X_2$. Assume that, for $i = 1, 2$, $X_i : \Omega_i \to S$, and that $\mathscr{F}_i$ is a σ-algebra on Ω_i, that $\mathscr{S}$ is a σ-algebra on S and that $\mathbb{P}_i$ is a probability measure on $\mathscr{F}_i$. Then convince yourselves that $X_1 \overset{(d)}{=} X_2$ means $\mathbb{P}_1 \circ X_1^{-1} = \mathbb{P}_2 \circ X_2^{-1}$.

2. If $Z_1, Z_2, \ldots$ are random elements and if τ is a random element of $\mathbb{N}$, what kind of random element is $(Z_1, \ldots, Z_\tau)$ (in which space does it take values)? What about $(Z_1, \ldots, Z_{\tau-1})$?
Hint: If the Z_i take values in a space S then $(Z_1, \ldots, Z_\tau)$ takes values in $S^* = \bigcup_{n \in \mathbb{N}} S^n$ if $\tau < \infty$ or in $S^{\mathbb{N}}$ if $\tau = \infty$. Suppose S comes with a σ-algebra $\mathscr{S}$. Then you should be able to attach the appropriate σ-algebra in the space of values of $(Z_1, \ldots, Z_\tau)$.

3. Prove the découpage de Lévy. Prove the simulation part of the découpage de Lévy.

4. Let $Z_1, Z_2, \ldots$ be a sequence of random elements forming a Markov process, that is, $(Z_1, \ldots, Z_{n-1}, Z_n)$ is independent of $(Z_n, Z_{n+1}, \ldots)$ conditional on Z_n. Let σ be a stopping time, that is, for each n we have $\mathbf{1}_{\{\sigma=n\}} = f_n(Z_1, \ldots, Z_n)$ for some measurable function f_n. Assume that $\mathbb{P}(\sigma < \infty) = 1$. Show that $(Z_1, \ldots, Z_{\sigma-1}, Z_\sigma)$ is independent of $(Z_\sigma, Z_{\sigma+1}, \ldots)$ conditional on Z_σ. In particular, prove the assertion in (1).

5. Let $X_1, X_2, \ldots$ be a sequence of random variables. Define the process reversed at t by
$$\widehat{X}_n := X_t - X_{t-n}, \quad 0 \le n \le t.$$
Let α be any stopping time for X_n, $n \ge 1$ and let $\alpha = \alpha_1 < \alpha_2 < \cdots$ be its iterates. Denote by $\alpha[t]$ the last occurrence of α_k such that $\alpha_k \le t$. We shall use the unorthodox notation $\alpha[X_1, \ldots, X_t; t]$ to denote the fact that this last occurrence is a certain deterministic function of $X_1, \ldots, X_t$. That is, given t and the piece $(X_1, \ldots, X_t)$ of the process there is a rule that tells us what the last occurrence is. This rule we denote by $\alpha[X_1, \ldots, X_t; t]$. We say that a stopping β is dual to α if, for each t,
$$\alpha[X_1, \ldots, X_t; t] + \beta[\widehat{X}_1, \ldots, \widehat{X}_t; t] = t.$$
Note that if β is dual to α then α is dual to β because $\widehat{\widehat{X}} = X$. In this sense, the first time k such that $X_k > 0$ and the first time k such that $X_k \le 0$ are dual of one another.

6. Prove the last equality in (2), namely that $\inf\{n > \alpha_k : X_n > X_{\alpha_k}\} = \inf\{n > \alpha_k : X_n > \overline{X}_{n-1}\}$, using simple logic.

7. Show the equality of the two expressions for $L(t)$ in (3). Namely, if $\alpha_1 < \alpha_2 < \cdots$ is a finite or infinite strictly increasing sequence of real numbers then $\sum_{k=1}^{\infty} \mathbf{1}_{\alpha_k \le t} = \sup\{k \ge 1 : \alpha_k \le t\}$.

8. Justify (5), i.e. that $\overline{X}_t = X_{\alpha_{L(t)}}$.

9. If $Z_1, Z_2, \ldots$ are random elements in, say, $\mathbb{R}$, if A is a Borel subset of $\mathbb{R}$, and if $\tau = \inf\{n \ge 1 : Z_n \in A\}$, then show that $Z_1, Z_2, \ldots$ is i.i.d. killed at rate $\mathbb{P}(\tau = \infty)$. (Convention: $\inf \varnothing = \infty$; so $\{\forall n \in \mathbb{N}\ Z_n \notin A\} = \{\tau = \infty\}$.)

10. Recall that the k-th cycle is defined as $\mathcal{C}(k) = (\xi_n, \delta_n,\ \alpha_{k-1} < n \le \alpha_k)$. We defined I to be the index of the first cycle containing a head. We defined $L(T)$ to be the last α_k that is $\le T$. Show (7): i.e. that $I = L(T) + 1$.
Solution: Recall that cycle $\mathcal{C}(k)$ has support $\alpha_{k-1} < n \le \alpha_k$. The index I is the index of the first cycle containing a head. But heads occur first at time $T+1$. Thus $\alpha_{I-1} < T+1 \le \alpha_I$ or $\alpha_{I-1} \le T < \alpha_I$. By (4) we have
$$L(T) = I - 1.$$

11. Draw the picture in (8).

12. Prove the last relation of Lemma 2: $t - \alpha_{L(t)} = \sup\{n \le t :\ \widehat{X}_n \le \underline{\widehat{X}}_{n-1}\}$.
Solution: If $I \subset [0,t]$ is empty then we define $\sup I = 0$ and $\inf I = t$. With this convention, we write
$$\begin{aligned}
\alpha_{L(t)} &= \sup\{1 \le n \le t :\ X_n > X_0, \ldots, X_{n-1}\} \\
&= \inf\{0 \le n \le t-1 :\ X_n \ge X_{n+1}, \ldots, X_t\} \qquad \textit{(by pure logic)} \\
&= \inf\{0 \le n \le t-1 :\ X_t - X_n \le X_t - S_{n+1}, \ldots, X_t - X_t\} \\
&= \inf\{0 \le n \le t-1 :\ \widehat{X}_{t-n} \le \widehat{X}_{t-n-1}, \ldots, \widetilde{X}_0\}
\end{aligned}$$
Hence
$$\begin{aligned}
t - \alpha_{L(t)} &= \sup\{1 \le t-n \le t :\ \widehat{X}_{t-n} \le \widehat{X}_{t-n-1}, \ldots, \widehat{X}_0\} \\
&= \sup\{1 \le n \le t :\ \widehat{X}_n \le \widehat{X}_{n-1}, \ldots, \widehat{X}_0\}.
\end{aligned}$$

13. Take the limit as $q \uparrow 1$ in Baxter's equations and thus derive a formula for $\mathbb{E}e^{i\theta X_\alpha}$ (interpreted as $\mathbb{E}(e^{i\theta X_\alpha}; \alpha < \infty)$)

14. Consider a symmetric random walk with continuous stationary-independent increments, that is, $\xi_1 \overset{(d)}{=} -\xi_1$ and $\mathbb{P}(\xi_1 = x) = 0$ for all $x \in \mathbb{R}$. Show that $\mathbb{E}q^\alpha = 1 - \sqrt{1-q}$. Find a formula for $\mathbb{P}(\alpha = n)$, for $n \in \mathbb{N} \cup \{\infty\}$.

15. Let X_n, $n = 1, 2, \ldots$, $X_0 = 0$, be a general random walk in $\mathbb{R}$. Let T be an independent geometric$_0$ random variable, $\mathbb{P}(T = n) = q^n p$, $n \ge 0$, let $\overline{X}_T = \max(X_0, \ldots, X_T)$ and let G_T be the last $n \le T$ such that $X_T = \overline{X}_T$ (the last time that the maximum is achieved; this could be unique, e.g., if the law of the increment is continuous). Derive the joint Laplace transform of $(G_T, \overline{X}_T)$: For $\eta, \theta \ge 0$,
$$\mathbb{E}e^{-\eta G_T - \theta \overline{X}_T} = \exp\left\{ \sum_{n\ge1} \int_{(0,\infty)} (e^{-\eta n - \theta x} - 1) n^{-1} q^n \mathbb{P}(X_n \in dx) \right\}$$
Hint: Let α be ANY stopping time and $L(t)$ the counting process of its iterates. From the "first to last" Proposition 1 aplied to the random walk (n, X_n) we have
$$\mathbb{E}z^{\alpha_{L(T)}} e^{i\theta X_{\alpha_{L(T)}}} = \frac{1 - \mathbb{E}q^\alpha}{1 - \mathbb{E}q^\alpha z^\alpha e^{i\theta X_\alpha}}.$$
Let α be THE stopping time $\inf\{n > 0 :\ X_n > 0\}$. Then $\alpha_{L(T)} = G_T$ and $X_{\alpha_{L(T)}} = \overline{X}_T$ (see (5) and Exercise 8). Hence the last display is precisely the joint transform of $(G_T, \overline{X}_T)$. Now use Baxter's equations (Proposition 2) in the right-hand side.

16. Prove Blumenthal's 0-1 law (11).
Hint: Use independent increments and Kolmogorov's 0-1 law: If $\xi_1, \xi_2, \ldots$ are independent random elements on a common probability space then the tail σ-algebra $\mathscr{T} = \bigcap_n \sigma(\xi_n, \xi_{n+1}, \ldots)$ is independent of itself and that every $A \in \mathscr{T}$ has probability 0 or 1.

17. Let X_t, $t \geq 0$, be a general Lévy process starting from $X_0 = 0$ and define $Q_t = X_t - \underline{X}_t$. Recall the times return $\mathfrak{r}$ and sojourn $\mathfrak{s}$ for Q: $\mathfrak{r} = \inf\{t > 0 : Q_t = 0\}$, $\mathfrak{s} = \inf\{t > 0 : Q_t > 0\}$. Show that $\mathfrak{r} = 0$ iff $\inf\{t > 0 : X_t \leq 0\} = 0$. This is a deterministic exercise. (It turns out that this holds iff $\int_0^\varepsilon t^{-1} \mathbb{P}(X_t \leq 0)\, dt = \infty$ for some, and hence all, $\varepsilon > 0$.)

18. Prove the decomposition Lemma 4.
Hint: For every $x \in U$ show that there is a largest open interval I_x included in U and containing x. Show that its endpoints are not in U. Then define $\mathscr{I} := \{I_x : x \in U\}$. Show that this works. Show that it is countable. Show that it is unique.

19. For an open set $U \subset \mathbb{R}$ let E be the (necessarily countable) set of endpoints of the components of U. Give an example of a set U such that E is a strict subset of U^c and such that U^c is nowhere dense.

20. An excursion of Q over a component (g, d) of the complement of the set $\overline{\mathfrak{Z}}$ is a random element. Of which space? See the analogy to Exercise 2.

21. Although it is obvious that, except in trivial cases, every path has a u-excursion, it is a little less obvious that there is a u-excursion for every path a.s. In other words, prove Lemma 5,
Hint: Uses the strong Markov property.

22. Prove Lemma 6.

23. Show that the right endpoints of all excursion intervals are stopping times (but that the left ones are not)–Lemma 7.

24. Show that, for $x < y$, $\mathbb{P}(N_x(g_1(y)) = k) = \left(1 - \frac{\Psi(y)}{\Psi(x)}\right)^k \frac{\Psi(y)}{\Psi(x)}$, $k = 0, 1, 2, \ldots$ and also show the independence claimed in Lemma 9.

25. Supply the details in the claim of Lemma 13 that L_ν^{-1}, $\nu \geq 0$, has independent increments. Show that, for $\nu, \mu \geq 0$, $L_{\nu+\mu}^{-1} - L_\nu^{-1} \perp\!\!\!\perp \mathscr{F}_{L_\nu^{-1}}$ a.s. on $L_\nu^{-1} < \infty$.

26. Show if a strong Markov process Q_t, $t \geq 0$, starts from some point x in its state space and remains at x for a positive amount of time σ then σ must be exponentially distributed and $Q_\sigma \neq x$ a.s. (The following example is instructive: If σ is an exponential random variable, the process $X_t^x := x + (t - \sigma \mathbf{1}_{x=0})^+$, $t \geq 0$, is Markov with state space $[0, \infty)$, but not strong Markov.)

27. Verify the second equation in (14) and the last one. The last one is due to the so-called Frullani integral. Theorem: let $f : (0, \infty) \to \mathbb{R}$ be measurable and integrable over any compact interval. Assume that $f(0) := \lim_{x \to 0} f(x)$ and $f(\infty) := \lim_{x \to \infty} f(x)$ exist. Then, for $a, b > 0$,
$$\int_0^\infty (f(ax) - f(bx)) x^{-1} dx = (f(0) - f(\infty)) \log(a/b).$$

Prove this under the additional assumption that f is continuously differentiable. To see why it should be true in general, use scale-invariance: The left-hand side is invariant under any map $(a, b) \mapsto (ta, tb)$ where t is a positive constant. Hence it is a function of a/b. Using measure-theoretic arguments, one can show that it is a measurable function and from this that it is a multiple of $\log(a/b)$.