# Self-Similar Markov Processes and SDEs

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# Goal

Along the example of self-similar Markov processes we try to obtain a better understanding of how to use generators and SDEs (with jumps).

Reminder from Andreas lectures without further details:
A pssMp can be restarted from 0 if and only if $\mathbb{E}[e^{\lambda\xi_1}] = 1$ for some $\lambda \in (0, 1/\alpha)$.

In these lectures we will go through SDEs, generators, etc. with the ultimative goal in mind to understand the „Cramér condition“ in the theorem.

Warning: We do NOT arrive at the most general results for ssMps but get to know a different point of view inspired by the SDE approach to CSBPs (or affine processes) due to (Bravo/Caballero/Lambert and Dawson/Li).

# Self-similar Markov processes (ssMp)

A strong Markov process $(X_t : t \geq 0)$ on $\mathbb{R}$ with RCLL paths, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}$, is a ssMp if there exists an index $\alpha \in (0, \infty)$ such that, for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x$$

is equal in law to

$$(X_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

nnssMp if sample paths are non-negative.

pssMp if sample paths are positve and absorbed at the origin.

# Content

- Examples of pssMps related to Brownian SDEs
- Lamperti's representation for pssMps and generators
- Lamperti SDE and jump diffusions (jump SDEs)

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# Several examples of ssMps

- Brownian motion $(B_t)$ is a ssMp with index 2
- stopped Brownian motion $(B_t 1_{(T_0>t)})$ is a pssMp with index 2
- Bessel processes of dimension $\delta$ - $Bes(\delta)$ - i.e. solutions of

$$dX_t = \frac{\delta - 1}{2}\frac{1}{X_t}\,dt + dB_t$$

  give pssMp with index 2 (modulus of Brownian motion).
- squared-Bessel processes of dimension $\delta$ - $Bes^2(\delta)$ - i.e. solutions of

$$dX_t = \delta\,dt + 2\sqrt{X_t}dB_t$$

  give pssMps with index 1.
- Brownian motion conditioned to be positive $(B^{\uparrow})$ is a pssMp with index 2 (and solves an SDE), called circle process by Andreas

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# How to check self-similarity?

There is no general approach!

- For $(B_t)$ show that scaled process is also a BM.
- For $(B_t 1_{(T_0>t)})$ consider the joint process $(B_t, \inf_{s\leq t} B_t)$.
- For the SDE examples use SDE theory.

# Self-similarity for $Bes^2(\delta)$ - the SDE way

$$
\begin{aligned}
cX_{tc^{-1}} &= c\left(X_0 + \delta tc^{-1} + \int_0^{tc^{-1}} 2\sqrt{X_s}\, dB_s\right) \\
&= cX_0 + \delta t + c\int_0^t 2\sqrt{X_{sc^{-1}}}\, d(B_{sc^{-1}}) \\
&= cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\, d(\sqrt{c}B_{sc^{-1}}) \\
&=: cX_0 + \delta t + \int_0^t 2\sqrt{cX_{sc^{-1}}}\, dW_t.
\end{aligned}
$$

Hence, $(X_t)$ and $(cX_{tc^{-1}})$ both satisfy the same SDE

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t$$

driven by some Brownian motions.

Why does this imply $Bes^2(\delta)$ is ssMp? $\rightarrow$ Need SDE uniqueness theory.

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# 1dim SDE Theory

Consider the 1dim SDE

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R},$$

driven by a BM.

## Notation (Solutions)

- A (weak) solution is a stochastic process satisfying almost surely the integral equation

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dB_s, \quad X_0 \in \mathbb{R}.$$

- A solution is called strong if it is adapted to the filtration generated by the driving noise $(B_t)$.

Reference e.g. Karatzas/Shreve

# 1dim SDE Theory

Question: Which SDEs can you solve explicitly?

Not many, only if Itō's formula helps. Remember: For $f \in C^2$ Itō's formula for semimartingales gives

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t \\ &= f'(X_t)a(X_t)dt + f'(X_t)\sigma(X_t)dB_t + \frac{1}{2}f''(X_t)\sigma^2(X_t)dt \end{aligned}$$

Example: Play around with the exponential function to solve

$$dX_t = aX_t dt + \sigma X_t dB_t.$$

Example: Which SDE is solved by $X_t = B_t^3$

$\rightarrow$ blackboard?

# 1dim SDE Theory

Consider the SDE

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R}.$$

## Notation (Uniqueness)

- We say weak uniqueness holds if any two weak solutions have the same law.
- We say pathwise uniqueness holds if any two weak solutions are indistinguishable.

Example: Tanaka's SDE

$$dX_t = sign(X_t)dB_t$$

has a weak solution, has no strong solution, weak uniqueness holds, pathwise uniqueness is wrong. *sign* is a bad function!

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# 1dim SDE Theory

**Theorem (Itō)**

If $a$ and $\sigma$ are Lipschitz continuous, then there is a unique strong solution.

Proof: Fixpoint theorem in good process space $\to$ constructive.

Problem: No interesting function is globally Lipschitz.

**Theorem**

If $a$ and $\sigma$ are locally Lipschitz and grow at most linearly, then there is a unique strong solution.

Problem: Many interesting functions are not locally Lipschitz.

**Theorem (Strook/Varadhan)**

If $a$ and $\sigma$ are bounded and continuous, then there is a weak solution.

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# 1dim SDE Theory

**Theorem**

Pathwise uniqueness implies weak uniqueness.

**Theorem**

Weak existence and pathwise uniqueness imply strong existence.

**Theorem**

Weak uniqueness implies strong Markov and Feller properties.

**Theorem (Yamada/Watanabe - Brownian case)**

If $a$ is locally Lipschitz and $\sigma$ is locally $\frac{1}{2}$-Hölder, then pathwise uniqueness holds.

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Note: Apply same strategy whenever you have an Itō formula!

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Remarks:

- To prove pathwise uniqueness there is a strategy!
- There is no general strategy to prove weak uniqueness!

This is a strange problem: Only know how to proceed in the harder case.

Note: Results (in law) extend to general stochastic equations (Kurtz).

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# Question

How would you construct a positive strong solution for

$$dX_t = \delta dt + 2\sqrt{X_t}dB_t, \quad X_0 = 0,$$

for $\delta > 0$? Note that

- $a \equiv \delta$ is Lipschitz
- $\sigma(x) = 2\sqrt{x}$ is $\frac{1}{2}$-Hölder

For $\delta \leq 0$ there is no positive solution (positive submartingales are absorbed at 0 or comparison theorem).

# A Counterexample

The SDE

$$dX_t = |X_t|^\beta dB_t, \qquad X_0 = 0,$$

has precisely one solution $X_t \equiv 0$ if $\beta \geq \frac{1}{2}$ ($\frac{1}{2}$-Hölder case).

For $\beta < \frac{1}{2}$ there are infinitely many solutions.

The equation has only one solution $X \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ (X_t)_{t \geq 0} : \int_0^\infty 1_{(X_s = 0)} \, ds = 0 \text{ a.s.} \right\}$$

Very hard, due to Bass/Burdy/Chen.

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# Another Counterexample

The Itō-Watanabe SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0, \tag{1}$$

has infinitely many solutions.

But: The SDE has only one positve solution in $\mathcal{S}$.

Question: Can you relate all solutions to the solutions in $\mathcal{S}$?

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# Back to self-similarity

Weak and pathwise uniqueness holds for

$$dX_t = \delta dt + 2\sqrt{X_t}dB_t,$$

hence $(X_t) \stackrel{\mathcal{L}}{=} (cX_{tc^{-1}})$, so $Bes^2(\delta)$ is a ssMp with index 1.

Remark: Same argument shows that the interesting positive solution to the Itō-Watanabe SDE is a ssMp. Alternatively, play with Itō-formula and use

**Lemma**

Suppose $(X_t)$ is a pssMp with index $\alpha$, then $(X_t^\alpha)$ is a pssMp with index 1.

Proof: Set $Y = X^\alpha$, then

$$\big(cY_{tc^{-1}}\big)_{t\geq 0} = \big((c^{1/\alpha}X_{tc^{-1}})^\alpha\big)_{t\geq 0} = \big((c^{1/\alpha}X_{t(c^{1/\alpha})^{-\alpha}})^\alpha\big)_{t\geq 0} = \big(Y_t\big)_{t\geq 0}$$

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# To remember for later

Solutions to

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t,$$

form a nnssMp that is NOT absorbed at zero if only if $\delta > 0$. Recall, a pssMp is by definition absorbed at 0.

# A discontinuous ssMp for later

## Definition

A Lévy process $(X_t)$ is called (strictly) $\alpha$-stable if it is also a self-similar Markov process.

- Theorem: $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- Theorem: Characteristic exponent $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + e^{-\pi i\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}\right), \qquad \theta \in \mathbb{R}.$$

  where $\rho = P_0(X_t \geq 0)$.
- Theorem: Assume jumps in both directions, then

$$\Pi(dx) = \left(\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho) 1_{\{x>0\}} + \sin(\pi\alpha\hat{\rho}) 1_{\{x<0\}}\right)\right) dx$$

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# Content

- Examples and Brownian SDEs
- Lamperti's representation for pssMps and generators
- Lamperti SDE and Jump Diffusions

# Notation

- Let $(\xi_t)$ a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time with rate in $q \in [0, \infty)$.
- Sometimes write $\xi^{(x)}$ if started in $x$, but always $\xi = \xi^{(0)}$.
- Define the integrated exponential Lévy process

$$I_t = \int_0^t e^{\alpha \xi_s} ds, \qquad t \geq 0,$$

and its limit $I_\infty := \lim_{t\uparrow\infty} I_t$.
- Define the inverse of the increasing process $I$:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0.$$

As usual, we work with the convention $\inf \emptyset = \infty$.

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# Notation

- Let $(\xi_t)$ a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time with rate in $q \in [0, \infty)$.
- Sometimes write $\xi^{(x)}$ if started in $x$, but always $\xi = \xi^{(0)}$.
- Define the integrated exponential Lévy process

$$I_t = \int_0^t e^{\alpha \xi_s} ds, \qquad t \geq 0,$$

  and its limit $I_\infty := \lim_{t \uparrow \infty} I_t$.
- Define the inverse of the increasing process $I$:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0.$$

  As usual, we work with the convention $\inf \emptyset = \infty$.

# Lamperti transform for pssMp

## Theorem (Part (i))

*If $X^{(x)}$, $x > 0$, is a pssMp with index $\alpha$, then it can be represented as follows. For $x > 0$,*

$$X_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \leq T_0,$$

*and $\xi$ is a (possibly killed) Lévy process.*

*Furthermore, $T_0 = x^\alpha I_\infty$, where $T_0 = \inf\{t > 0 : X_t^{(x)} \leq 0\}$.*

Note: Using $\xi^{(\log x)} = \xi + \log x$, one can also write

$$X_t^{(x)} = \exp\{\xi^{(\log x)}_{\varphi(t)}\}, \qquad t \leq T_0.$$

Note: First version more common, but second version shows better what happens.

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Note: First version more common, but second version shows better what happens.

# Lamperti transform for pssMp

**Theorem (Part (ii))**

*Conversely, suppose $\xi$ is a given (possibly killed) Lévy process. For each $x > 0$, define*

$$\boxed{X_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}\mathbf{1}_{(t<x^\alpha I_\infty)}, \qquad t \geq 0.}$$

*Then $X^{(x)}$ defines a pssMp, up to its absorption time at the origin.*

# Lévy reminder

For a Lévy process $\xi$ either

(0) $\xi$ is killed

(a) $\lim_{t\uparrow\infty} \xi_t = +\infty$ a.s.

(b) $\lim_{t\uparrow\infty} \xi_t = -\infty$ a.s.

(c) $\limsup_{t\uparrow\infty} \xi_t = \infty$, $\liminf_{t\uparrow\infty} \xi_t = -\infty$ a.s.

## Definition

We say

(0) $\xi$ is killed

(a) $\xi$ drifts to $+\infty$

(b) $\xi$ drifts to $-\infty$

(c) $\xi$ oscillates

Example: $\xi_t = at + B_t$, only depends on $a$

# Consequenes for pssMp

## Consequence for pssMps

For all $x > 0$ we have

(1) $T_0 = \infty$ a.s. iff $\xi$ drifts to $+\infty$ or oscillates,

(2) $T_0 < \infty$ and $X^{(x)}_{T_0-} = 0$ a.s. iff $\xi$ drifts to $-\infty$,

(3) $T_0 < \infty$ and $X^{(x)}_{T_0-} > 0$ a.s. iff $\xi$ is killed.

$\rightarrow$ blackboard drawings

# Summary

$$(X, \mathrm{P}_x)_{x>0} \text{ pssMp} \qquad \leftrightarrow \qquad (\xi, \mathbb{P}_y)_{y\in\mathbb{R}} \text{ killed Lévy}$$

$$X_t = \exp(\xi_{S(t)}), \qquad \qquad \xi_s = \log(X_{T(s)}),$$

$S$ a random time-change $\qquad$ $T$ a random time-change

$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \\ X \text{ has continuous paths} \end{array}\right\} \quad \leftrightarrow \quad \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \\ \xi \text{ has continuous paths} \end{array}\right.$$

# Example

We know $Bes^2(\delta)$

$$dX_t = \delta dt + 2\sqrt{X_t}dB_t$$

is self-similar so it is a pssMp up to $T_0$. One can prove in many ways that $Bes^2(\delta)$ hits zero (continuously) if and only if $\delta < 2$.

Questions: Can you guess the corresponding Lévy process and why $\delta < 2$? (there are not many choices)

$\to$ Let's find $\xi$ with the help of generators.

# Generator Theory

<u>Definition:</u> The generator of a Markov process on $\mathcal{X}$ is the linear operator

$$\mathcal{A}f(x) = \lim_{t\to 0} \frac{E^x[f(X_t)] - f(x)}{t}, \quad x \in \mathcal{X},$$

defined on the domain $\mathcal{D}(\mathcal{A}) = \{f \in C_b : \mathcal{A}f(x) \text{ exists in } C_b\}$.

<u>Note:</u> It is normal to know the action $\mathcal{A}$ but not the full domain $\mathcal{D}(\mathcal{A})$. To know a large (dense) subset of $\mathcal{D}(\mathcal{A})$ is typically enough to work with the generator (see Ethier/Kurtz book).

<u>Note:</u> Often one uses generator computations (without caring for the domain) to guess the results and then develop different proof.

## Dynkin Formula and it's inverse to compute the generator

(1) If $(A, \mathcal{D}(\mathcal{A}))$ is the generator of $(X_t)$ and $f \in \mathcal{D}(\mathcal{A})$, then

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds, \quad t \geq 0,$$

is a martingale.

(2) If $f \in C_b$ and there is $g \in C_b$ with

$$M_t = f(X_t) - f(X_0) - \int_0^t g(X_s)\, ds, \quad t \geq 0,$$

is a martingale, then $f \in \mathcal{D}(\mathcal{A})$ and $g = \mathcal{A}f$.

Finding a process so that the righhand sides are martingales is also called „solving the martingale problem“ for $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. Such a process has generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$.

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# Some examples (without exact domain)

- For solutions of $dX_t = a(X_t)dt + \sigma(X_t)dB_t$ the generator acts as

$$\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad x \in \mathbb{R},$$

because (Itō formula)

$$\begin{aligned} f(X_t) - f(X_0) - \int_0^t f'(X_s)a(X_s)ds - \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)ds \\ = \int_0^t f'(X_s)\sigma(X_s)dB_s, \quad t \geq 0. \end{aligned}$$

and Dynkin formula (when is an Itō-integral a martingale?).

- For a Lévy process with triplet $(a, \sigma^2, \Pi)$ the generator acts as

$$\begin{aligned} \mathcal{A}f(x) &= af'(x) + \frac{1}{2}\sigma^2 f''(x) \\ &\quad + \int_{\mathbb{R}} \left(f(x+u) - f(x) - f'(x)u\mathbf{1}_{|u|\leq 1}\right)\Pi(du), \quad x \in \mathbb{R}. \end{aligned}$$

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# Some examples (with domain)

- $(\frac{1}{2}\Delta, C_0(\mathbb{R}))$ generates Brownian motion $B$ on $\mathbb{R}$
- $$\left(\frac{1}{2}\Delta, C_0(\mathbb{R}_+) \cap \{f : f(0) = 0\}\right)$$

  generates Brownian motion absorbed at zero $B^\dagger$.

# The usefullness of generators

Many transformations of Markov processes can be understood with „formal" computations of generators. This allows to guess results, rigorous proofs often with other techniques.

Example: Conditioning a process to avoid a set $B$ often leads to an $h$-transform with a positive harmonic function which vanishes at $B$. For $h$-transforms the formula $\mathcal{A}^h f(x) = \frac{1}{h(x)} \mathcal{A} fh(x)$ holds.

Some might know that $(B_t^{\uparrow})$, BM conditioned to be positive is an $h$-transform of $(B_t^{\dagger})$ with $h(x) = x$. Pluging-in gives

$$
\begin{aligned}
\mathcal{A}^{\uparrow} f(x) &= \frac{1}{h(x)} \frac{1}{2} \frac{d^2}{dx^2} fh(x) \\
&= \frac{1}{h(x)} \frac{1}{2} (f''(x)h(x) + f(x)h''(x) + 2f'(x)h'(x)) \\
&= \frac{1}{2} f''(x) + \frac{1}{x} f'(x)
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$\rightarrow$ $(B_t^{\uparrow})$ is $Bes(3)$-process, self-similar with index 1. Stable case harder!

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# Generators and time-change

Time-Change: Two Markov processes $X$ and $\tilde{X}$ with generators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ satisfy

$$\boxed{\mathcal{A}f(x) = \beta(x)\tilde{\mathcal{A}}f(x), \quad x \in \mathcal{X},}$$

for a measurable function $\beta : \mathcal{X} \to \mathbb{R}$ if and only if

$$\boxed{X_t = \tilde{X}_{(\int_0^t \beta^{-1}(\tilde{X}_s)\,ds)^{-1}}, \quad t \geq 0,}$$

and

$$\inf\left\{t : \int_0^t \beta^{-1}(\tilde{X}_s)\,ds = \infty\right\} = \inf\{t : \beta(\tilde{X}_t) = 0\}.$$

Theorem due to Volkonskii. (Proof: Martingale problem, change variables).

**Note: Multiplication in generator changes only speed not directions.**

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**<u>Note:</u> Multiplication in generator changes only speed not directions.**

# A fun example

Fun example: SABR model (stochastic $\alpha, \beta, \rho$ model with $\beta < 1$)

$$\begin{cases} dX_t = \sigma_t X_t^{\beta} dB_t \\ d\sigma_t = \alpha\, \sigma_t dW_t \end{cases}$$

Suppose $B$ and $W$ are independent even though the $\rho$ in the name stands for their correlation.

Question: Any idea for the limit $\lim_{t\to\infty} X_t$?

Hint: Generator is

$$\mathcal{A}f(x,y) = y^2 \left( x^{2\beta} \frac{1}{2} f_{xx}(x,y) + \frac{\alpha^2}{2} f_{yy}(x,y) \right).$$

# Lamperti's representation now seen through generators

**Theorem (Lamperti), continuous case**

The action of the generator for a continuous pssMp is

$$\mathcal{A}f(x) = \frac{1}{x^\alpha}\left[\left(a + \frac{\sigma^2}{2}\right) x\, f'(x) + \sigma\, x^2\, f''(x)\right]$$

and the corresponding Lévy process is $\xi_t = at + \sigma B_t$.

<u>Why?</u> Righthand side is

$$\frac{1}{x^\alpha}\mathcal{A}_{e^{\text{BM with drift}}} f(x),$$

where $\mathcal{A}_{e^{\text{BM with drift}}}$ is the generator of $e^{\text{BM with drift}}$ and you know which SDE it solves.

# All continuous pssMps

Rewritten, all generators of continuous pssMps have action

$$\mathcal{A}f(x) = \left(a + \frac{\sigma^2}{2}\right) x^{1-\alpha} f'(x) + \frac{\sigma^2}{2} x^{2-\alpha} f''(x)$$
$$= \log \mathbb{E}[e^{\xi_1}] x^{1-\alpha} f'(x) + \frac{\sigma^2}{2} x^{2-\alpha} f''(x).$$

Setting $\delta = a + \frac{\sigma^2}{2}$, all pssMps with continuous paths and index $\alpha$ are solutions (up to $T_0$) to

$$dX_t = \delta X_t^{1-\alpha} dt + \sigma X_t^{1-\alpha/2} dB_t, \qquad X_0 > 0, \tag{2}$$

for some $\delta \in \mathbb{R}, \sigma > 0$.

Corollary: Solutions to the SDE (2) hit zero in finite time a.s. if $\delta < \frac{\sigma^2}{2}$. Otherwise, almost surely zero is not hit.

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# $Bes^2(\delta)$ again

With $\alpha = 1$ and $\sigma = 2$ we get back to $Bes^2(\delta)$:

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t$$

with corresponding Lévy process

$$\xi_t = (\delta - 2)t + 2B_t.$$

Hence, zero is hit in finite time iff $\delta < 2$ because this is when $\xi$ drifts to $-\infty$.

# Lamperti's representation now seen through generators

**Theorem (Lamperti), for $E[e^{\xi_1}] < \infty$**

The action of the generator for a pssMp is

$$
\begin{aligned}
&\mathcal{A}f(x) \\
&= \frac{1}{x^\alpha}\bigg[\log E[e^{\xi_1}]\, x\, f'(x) + \frac{\sigma^2}{2}\, x^2\, f''(x) \\
&\quad + \int_{-\infty}^{\infty} \big[f(e^u x) - f(x) - f'(x)x(e^u - 1)1_{|u|\leq 1}\big]\Pi(du)\bigg]
\end{aligned}
$$

and the corresponding Lévy process has triplet $(a, \sigma^2, \Pi)$.

<u>Why?</u> Righthand side is

$$\mathcal{A}f(x) = \frac{1}{x^\alpha}\mathcal{A}_{e^\xi} f(x),$$

where $A_{e^\xi}$ is the generator of $e^\xi \to$ blackboard

# General Remarks 1

There are three transformations for Markov processes and we know what happens:

- change space (Itō formula)
- change time (Volkonskii)
- reverse time ($h$-transform)

Keep this in mind if you want to analyze a process !!!

# General Remarks 2

For pssMps (and other processes such as CSBPs) there are three equivalent ways of thinking:

- time-change (i.e. Lamperti representation)
- generator
- SDE $\rightarrow$ last chapter

All have advantages and disadvantages. Advantages are

- time-change can be good to analyze asymptotics
- generator good for quick calculations
- SDE good because you have Itō formula, local times for instance, etc.

Time-change has crucial disadvantage: No way to analyse process after hitting zero, entire path of the Lévy process is already used.

# Content

- Examples and Brownian SDEs
- Lamperti's representation for pssMps and generators
- Lamperti SDE and Jump Diffusions

# Extending pssMps to initial value 0 - possible or not?

- Recurrent case (continuous exit)

  $\rightarrow$ blackboard
- Transient case

  $\rightarrow$ blackboard

Next: Simple proof for special case of spec negative pssMps, assume $\alpha = 1$.

# Lévy SDEs

A Lévy SDE

$$dX_t = a(X_t)dt + \sigma(X_{t-})dL_t$$

driven by a Lévy process is an abbreviation for

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t \sigma(X_{s-})dL_s, \quad t \geq 0.$$

Definitions and results mostly analogous to Brownian theory, similar Itō construction of stochastic integral.

Example: If $(L_t)$ is spec pos $\alpha$-stable, then pathwise uniqueness holds if $a$ is Lipschitz and $\sigma$ is $(1-\frac{1}{\alpha})$-Hölder (Li/Mytnik). If $(L_t)$ is symmetric, then $1-\frac{1}{\alpha}$ changes to $\frac{1}{\alpha}$.

# Jump diffusions

We want more general equations in light of the Lévy-Itō representation of Lévy processes:

$$
\begin{aligned}
X_t = X_0 &+ \int_0^t a(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s \\
&+ \int_0^t \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,du) + \int_0^t \int_V d(X_{s-},u)\mathcal{M}(ds,du)
\end{aligned}
$$

where

- $\mathcal{N}$ PPP on $[0,\infty)\times U$ with intensity $\mathcal{N}'(ds,du) = ds\nu(du)$ and $\nu$ is $\sigma$-finite
- $\mathcal{M}$ PPP on $[0,\infty)\times V$ with intensity $\mathcal{M}'(ds,du) = ds\mu(du)$ and $\mu$ is finite

# Jump Diffusions

Integrals defined as in the Lévy-Itō representation of Lévy processes:

$$\begin{aligned}
&\int_0^t \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,du)\\
&\overset{L^2}{:=} \lim_{\varepsilon\to 0} \int_0^t \int_{U_\varepsilon} c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,du)\\
&:= \lim_{\varepsilon\to 0} \Big( \int_0^t \int_{U_\varepsilon} c(X_{s-},u)\mathcal{N}(ds,du) - \int_0^t \int_{U_\varepsilon} c(X_{s-},u)\mathcal{N}'(ds,du)\Big)\\
&= \lim_{\varepsilon\to 0} \Big( \sum_{x\in\mathcal{N}([0,t]\times U_\varepsilon)} c(X_{s-},x) - \int_0^t \int_{U_\varepsilon} c(X_{s-},u)\,\nu(du)\,ds\Big).
\end{aligned}$$

Warning: In general both limits can be infinite but the compensated integral converges under suitable conditions.

Note: If limiting compensator integral is finite, then jump integral is finite and integral is difference of jump integral and compensator integral.

# Jump diffusions

Example 1:
Lévy processes in Lévy-Itō form are jump diffusions:

- $U = [-1, 1]$, $\mathcal{N}'(ds, du) = ds\, \Pi(du)$,
- $V = [-1, 1]^c$, $\mathcal{M} = \mathcal{N}$,
- $a(x) = a$, $\sigma(x) = \sigma$, $c(x, u) = u$, $d(x, u) = u$.

Example 2: Lévy SDE written as special jump diffusion:

- $U = [-1, 1]$, $\mathcal{N}'(ds, du) = ds\, \Pi(du)$,
- $V = [-1, 1]^c$, $\mathcal{M} = \mathcal{N}$,
- $c(x, u) = c(x)u$, $d(x, u) = d(x)u$.

Jumps of Lévy SDE at $t$ take the form $d(X_{t-})\Delta L_t$ just as a Brownian integral gives $\sigma(X_t)\Delta B_t$.

General jump diffusions have jumps $d(X_{t-}, \Delta L_t)$ which is more flexible for modelling.

# Itō formula for jump diffusions

With $X$ as above and $f \in C^2$, we get

$$
\begin{aligned}
& f(X_t) - f(X_0) \\
&= \int_0^t f'(X_s)a(X_s)\,ds + \int_0^t f'(X_s)\sigma(X_s)\,dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\,ds \\
&\quad + \int_0^t \int_U \big[f(X_{s-} + c(X_{s-},u)) - f(X_{s-})\big](\mathcal{N} - \mathcal{N}')(ds,du) \\
&\quad + \int_0^t \int_V \big[f(X_{s-} + d(X_{s-},u)) - f(X_{s-})\big]\mathcal{M}(ds,du) \\
&\quad + \int_0^t \int_U \big[f(X_s + c(X_s,u)) - f(X_s) - f'(X_s)c(X_s,u))\big]\mathcal{N}'(ds,du).
\end{aligned}
$$

Special case: Lévy for $a = \sigma = const$ and $d(x,u) = c(x,u) = u$ and $\mathcal{N} = \mathcal{M}$.

If $f$ is bounded, then

$$\int_0^t \int_V \left| f(X_{s-} + c(X_{s-}, u)) - f(X_{s-}) \right| \mu(du)\, ds$$
$$\leq 2||f||_\infty t \int_V \mu(du) < \infty$$

so adding and substracting compensation for $\mathcal{M}$ gives

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A} f(X_s)\, ds$$

is a martingale, where

$$\begin{aligned}\mathcal{A} f(x) = {} & a(x) f'(x) + \frac{1}{2}\sigma^2(x) f''(x) \\ & + \int_U \big[ f(x + c(x,u)) - f(x) - f'(x) c(x,u) \big] \nu(du) \\ & + \int_V \big[ f(x + d(x,u)) - f(x) \big] \mu(du).\end{aligned}$$

Consequence: Know generator action for jump diffusions just as for SDEs.

If $f$ is bounded, then

$$\int_0^t \int_V \left| f(X_{s-} + c(X_{s-}, u)) - f(X_{s-}) \right| \mu(du)\, ds$$
$$\leq 2||f||_\infty t \int_V \mu(du) < \infty$$

so adding and substracting compensation for $\mathcal{M}$ gives

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# Jump diffusions

Remark: All general weak SDE theorems hold more or less equally for jump diffusions. Only pathwise uniqueness results need adjustment.

Remark: There are some pathwise uniqueness results, essentially same proof as for BM (Itō formula with $\phi_n(\cdot) \to |\cdot|$). More difficult because of unfriendly jump Itō formula.

# Jump diffusions and time-change

Suppose solution $\tilde{X}$ of a jump diffusion has generator $\tilde{\mathcal{A}}$. How to produce time-change $X$ with generator $\mathcal{A} = \beta\tilde{\mathcal{A}}$?

We know how to change drift and diffusion, but what to do with the jumps? $\rightarrow$ add extra component in PPP!

$$\begin{aligned} X_t = X_0 &+ \int_0^t \beta(X_s)a(X_s)\,ds + \int_0^t \sqrt{\beta(X_s)}\,\sigma(X_s)\,dB_s \\ &+ \int_0^t \int_0^{\beta(X_{s-})} \int_U c(X_{s-},u)(\mathcal{N}-\mathcal{N}')(ds,dr,du) \\ &+ \int_0^t \int_0^{\beta(X_{s-})} \int_V d(X_{s-},u)\mathcal{M}(ds,dr,du), \end{aligned}$$

where

- $\mathcal{N}$ PPP on $[0,\infty)\times[0,\infty)\times U$ with $\mathcal{N}'(ds,dr,du) = ds\,dr\,\nu(du)$
- $\mathcal{M}$ PPP on $[0,\infty)\times[0,\infty)\times V$ with $\mathcal{M}'(ds,dr,du) = ds\,dr\,\mu(du)$

Exercise: Calculate generator for $X$ with Itō formula to confirm $\mathcal{A} = \beta\tilde{\mathcal{A}}$.

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# Exercise

Please find an SDE representation for pssMps with $\alpha = 1$ !

# Lamperti SDE

## Theorem (Barczy, D.)

Every pssMp with $\alpha = 1$ can be written as solution to

$$X_t = X_0 + \left( a + \frac{\sigma^2}{2} + \int_{\{|u|\le 1\}} (e^u - 1 - u)\,\Pi(\mathrm{d}u) \right) t + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s$$
$$+ \int_0^t \int_0^{1/X_{s-}} \int_{\{|u|\le 1\}} X_{s-}[e^u - 1](\mathcal{N} - \mathcal{N}')(\mathrm{d}s, dr, \mathrm{d}u)$$
$$+ \int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\mathcal{N}(\mathrm{d}s, dr, \mathrm{d}u), \quad t \le T_0,$$

where $(a, \sigma^2, \Pi)$ is a Lévy triplet and

- $B$ is a BM
- $\mathcal{N}$ is a PPP on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ with intensity $ds \otimes dr \otimes \Pi(du)$

# Lamperti SDE

The equation is not very nice at first glance.

But:

- If we assume $E[e^{\xi_1}] < \infty$ we learn something.
- If we assume $\xi$ is spec neg, we can do everything we wish.

# Lamperti SDE

If $E[e^{\xi_1}] < \infty$, then

$$\begin{aligned}
&\int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\mathcal{N}'(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u) \\
&= \int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\, ds\, dr\, \Pi(du) \\
&= t \int_{\{|u|>1\}} [e^u - 1]\Pi(du) < \infty,
\end{aligned}$$

hence,

$$\begin{aligned}
&\int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1]\mathcal{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u) \\
&= \int_0^t \int_0^{1/X_{s-}} \int_{\{|u|>1\}} X_{s-}[e^u - 1](\mathcal{N} - \mathcal{N}')(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u) \\
&+ t \int_{\{|u|>1\}} [e^u - 1]\Pi(du).
\end{aligned}$$

# Lamperti SDE

Using Lévy-Khintchin

$$\log E[e^{\xi_1}] = a + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^u - 1 - u 1_{\{|u| \leq 1\}})\, \Pi(du)$$

we can simplify the SDE to

$$\boxed{\begin{aligned} X_t = X_0 + \log E[e^{\xi_1}] t + \sigma \int_0^t \sqrt{X_s} \mathrm{d}B_s \\ + \int_0^t \int_0^{1/X_{s-}} \int_{\mathbb{R}} X_{s-}[e^u - 1](\mathcal{N} - \mathcal{N}')(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u). \end{aligned}}$$

Note: Call both SDEs Lamperti SDE because they are equivalent to Lamperti's representation.

# Lamperti SDE in the spectrally negative case

**Theorem (Barczy, D.)**

- Pathwise uniqueness holds for the Lamperti SDE.
- Precisely for $\log E[e^{\xi_1}] > 0$ there are strong solutions for all $X_0 \geq 0$ to the Lamperti SDE and pathwise uniqueness holds.
- Solutions are self-similar Markov processes with non-negative paths.

Proof: Ugly Yamada/Watanabe type arguments.

# Lamperti SDE

**Theorem**

The self-similar recurrent extensions of Fitzsimmons, Rivero and also the limit laws $P^0$ of Bertoin, Caballero, Chaumont, Kyprianou, Pardo, Rivero, Savov, ... are solutions to the SDE.

Proof: As above for $Bes^2(\delta)$: Show that $(X_t)$ and $(cX_{tc^{-1}})$ solve same equation, then use uniqueness.

Exercise: Please proof uniqueness also for positive jumps!

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# Lamperti SDE

Why is Lamperti SDE special?

- Lamperti SDE for $t \leq T_0 \quad \Longleftrightarrow \quad$ Lamperti's representation.
- Lamperti's respresentation does not work immediately for $t > T_0$.
- BUT: Lamperti SDE works immediately for $t > T_0$ iff the necessary and sufficient condition is fulfilled.

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# Summary

We discussed definitions, examples and connections for

- time-change
- generators
- SDEs

In some sense those are equivalent, but approaches have different advantages.

For pssMps we discussed

- time-change representation
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For pssMps the SDE representation has a magic feature: Can be extended after hitting zero.

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